A brief Introduction to Dynamical Systems and Chaos Theory
Fundamentals of Analyzing Biomedical Signals

Dynamical Systems
Motivation

**mathematics:**

**dynamical system:** is a system in which a *function* describes the *time dependence* of a point in some geometrical space.

**physics:**

**dynamical system:** is described as a particle (or an ensemble) whose *state varies over time*; obeys differential equations involving time derivatives.

Making predictions about the system's future behavior requires an analytical solution of such equations or their integration over time through computer simulation.
Continuous-time dynamical systems

ordinary differential equation (first order)

\[ \mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^d : \quad \frac{d\mathbf{x}(t)}{dt} = f(t, \mathbf{x}(t), \beta) \]

**terminology**

\( \mathbf{x}(t) \) \quad \text{state}

\( \mathbf{x}(0) = \mathbf{x}(t_0) \) \quad \text{initial state/condition}

\( x_1, x_2 \ldots, x_d \) \quad \text{state variables}

\( d \) \quad \text{dimension of system}

\( \beta \) \quad \text{control parameter}

\( f \) \quad \text{nonlinear function (in case of nonlinear system)}

\( \frac{\partial f}{\partial t} = 0 \) \quad \text{autonomous system}

\( \frac{\partial f}{\partial t} \neq 0 \) \quad \text{driven system}
Continuous-time dynamical systems

ordinary differential equation (first order)

\[ \mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^d ; \quad \frac{d\mathbf{x}(t)}{dt} = f(t, \mathbf{x}(t), \beta) \]

- given by some model
- find solution for specific initial condition (given control parameter)
  often only possible numerically
  often not of particular interest
- allows general statements about ensembles of solutions

- extensions: delay, stochastic, partial differential equations
Complexity of dynamical systems

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.

Robert M. May
Simple mathematical models with very complicated dynamics, Nature 261 (1976)

some elementary examples in physics:
- classical pendulum / harmonic oscillator
- driven and damped pendulum
- celestial mechanics
Example: Lottka-Volterra Model (predator-prey dynamics)

\[
\begin{align*}
\frac{dx}{dt} &= a_1 x - a_2 xy \\
\frac{dy}{dt} &= -a_3 y + a_4 xy
\end{align*}
\]

- \(x\): number of prey (e.g. hare)
- \(y\): number or predator (e.g. lynx)
- \(a_1, \ldots, a_4\): positive real parameter describing interaction between species

Observation

Model data

from: www.scholarpedia.org/article/Predator-prey_model
Example: Lorenz oscillator

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= x(\rho - z) - y \\
\frac{dz}{dt} &= xy - \phi z
\end{align*}
\]

- simple model for atmospheric convection
- butterfly effect:
  does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?

from: Kumar et al., Clim. Dyn. 38, 1521, 2012

from: G. Ansmann
Chaotic dynamics (tentative definition)

- sensitive to initial conditions (butterfly effect)

- qualitatively recurring

*but not:*

- periodic

- stagnant

- “escalating”
Discrete-time dynamical system

iterative map

\[ x_t \in \mathbb{R}^d ; \quad x_{t+1} = F(t, x_t, \beta) \]

- easier to analyze
- easier to simulate
- every ODE can be transformed to a map
  (e.g., via numerical integration or Poincaré sections)
Example: logistic map

model for population growth (1837)

$$x_{n+1} = r x_n (1 - x_n); \quad x_n \in [0, 1]; \quad r \in [0, 4]$$
Example: logistic map
Example: logistic map
Example: logistic map

The Feigenbaum constant:

Let \( r_1, r_2, \ldots \) denote the values of control parameter \( r \) at which bifurcations happen. We find:

\[
\lim_{i \to \infty} \frac{r_i - r_{i-1}}{r_{i+1} - r_i} = \delta \approx 4.669
\]

- \( \delta \): Feigenbaum constant
- universal for many similar processes
- also found in natural systems
  - turbulent cascade in fluids
  - nonlinear oscillations in electric circuits
  - nonlinear oscillations in chemical reactions (Belousov-Zhabotinskii reaction)
  - heart: ventricular fibrillation (lethal)
Phase space

ordinary differential equation (first order)

\[ x : \mathbb{R} \rightarrow \mathbb{R}^d ; \quad \frac{dx(t)}{dt} = f(t, x(t), \beta) \]

**Phase space:**
representation of states/trajectories \((x)\) of the dynamics in \(d\)-dimensional space.

- time is only implicit
- also known as state space

Example:
harmonic oscillator
Invariant sets

**Time-evolution function:**

\[ \phi_{f,\tau} : \mathbb{R}^d \to \mathbb{R}^d ; \phi_{f,\tau}(x(t)) = x(t + \tau) \]

for any solution \( x \) of \( \frac{dx(t)}{dt} = f(x(t), \beta) \) analogously for maps

**Forward-invariant set / manifold:**

\( S_f \subset \mathbb{R}^d \) is a non-empty set for which holds:

\[ \forall \tau > 0 : S_f = \phi_{f,\tau}(S_f) := \{ \phi_{f,\tau}(x) | x \in S \} \]

**Irreducible invariant set:**

An invariant set without an invariant true subset
Attractors

$\mathcal{A}_f \subset \mathbb{R}^d$ denotes a set for which holds:

- $\mathcal{A}_f$ is an irreducible forward-invariant set
- there exists a neighborhood $\mathcal{B}_{\mathcal{A}_f} \supset \mathcal{A}_f$ such that
  \[
  \lim_{\tau \to \infty} \phi_{f,\tau}(\mathcal{B}_{\mathcal{A}_f}) = \mathcal{A}_f
  \]
- the maximal $\mathcal{B}_{\mathcal{A}_f}$ is called \textit{basin of attraction}
- dynamics within $\mathcal{B}_{\mathcal{A}_f} \cap \mathcal{A}_f$, i.e., motion onto the attractor, is called \textit{transient}
- more elaborate definitions to handle pathological cases
Attractors

- if the system dynamics is confined to a certain region in phase space, then this region is called *attractor*

- set of all solutions of the system’s dynamical equations

- three kinds of (irreducible) invariant sets / attractors important to this course:
  - fixed points
  - periodic
    - simple periodic / limit cycle
    - quasiperiodic / torus, hypertorus
  - strange / chaotic / fractal

Each type corresponds to a different type of dynamics
Intermezzo: Quasiperiodicity

Incommensurability

two numbers $a$ and $b$ are incommensurable iff $\frac{a}{b} \notin \mathbb{Q}$

Quasiperiodicity

superposition / combination of two (or more) periodic processes with incommensurable frequencies

from: G. Ansmann
Intermezzo: Attractor Basins

magnetic pendulum
Intermezzo: Attractor Basins

Dynamical Systems

Julia sets
Characterizing fixed points

for continuous, differentiable maps \((x_{t+1} = F(t, x_t, \beta))\), we have:

\[ z \text{ is fixed point of } F \iff \{z\} \text{ is invariant set} \iff F(z) = z \]

from local linearization and continuity of \(F\), we have:

\[ z \text{ is stable fixed point of } F \iff \{z\} \text{ is attractor} \iff F(z) = z \text{ and } |F'(z)| < 1 \]
Characterizing fixed points

for ordinary differential equations \( \frac{dx(t)}{dt} = f(t, x(t), \beta) \), we have:

\[ z \text{ is fixed point of } f \iff \{z\} \text{ is invariant set} \iff f(z) = 0 \]

from local linearization and continuity of \( f \), we have:

\[ z \text{ is stable fixed point of } f \iff \{z\} \text{ is attractor} \iff f(z) = 0 \text{ and } \nabla f \text{ has no eigenvalue } \mu \text{ with } \Re(\mu) > 0 \]
Poincaré sections

limit cycle:  
→ point

torus:  
→ limit cycle

chaotic motion:  
→ complex structures
Poincaré sections

Let $s_0 < s_1 < \ldots$ denote the times of “marker events”, e.g.:

- intersections of the trajectory with a plane (in a given direction)
- local extrema of some observable
- given phase of driving oscillation

**Poincaré map**

the map that yields the sequence $x(s_0), x(s_1), \ldots$

- may distill relevant aspects of the dynamics
- simplifies dynamics
- allows to estimate stability of periodic solutions
- may be difficult to determine
Classifying dynamical systems via divergence

consider $f$ as a vector field in phase space

$\rightarrow f$ describes the phase-space flow

$\rightarrow \nabla f$ describes expansion/contraction of infinitesimal phase-space volume $V(t)$ under $f$, i.e., time evolution (Liouville’s theorem):

$$\frac{d}{dt} |V(t)| = |V(t)| \nabla \cdot f(V(t))$$
Classifying dynamical systems via divergence

Three kinds of dynamics:

<table>
<thead>
<tr>
<th>divergence</th>
<th>name</th>
<th>attractors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \cdot f = 0$</td>
<td>conservative</td>
<td>no (Liouville)</td>
</tr>
<tr>
<td>$\nabla \cdot f &gt; 0$</td>
<td>unstable</td>
<td>no</td>
</tr>
<tr>
<td>$\nabla \cdot f &lt; 0$</td>
<td>dissipative</td>
<td>yes</td>
</tr>
</tbody>
</table>

(assuming constant sign along the trajectory)
Divergence and Lyapunov exponents

• divergence quantifies growth of volumes

\[ V(t) = V(t_0) e^{(\nabla \cdot f)t} \]

• Lyapunov exponents quantify growth of “vectors”
Lyapunov exponents

Example: two identical Lorenz oscillators with initial conditions; one oscillator is slightly perturbed ($10^{-14}$) at $t = 30$

de error grows exponentially

growth limited by system size

from: G. Ansmann
Largest Lyapunov exponent

Consider evolution of two nearby trajectory segments $s_1$ and $s_2$

For infinitesimally close trajectory segments $|s_1(t) - s_2(t)| \rightarrow 0$ and for infinite time evolution ($\tau \rightarrow \infty$) the distance between segments grows or shrinks exponentially:

$$|s_1(t + \tau) - s_2(t + \tau)| = |s_1(t) - s_2(t)| e^{\lambda_1 \tau}$$
Largest Lyapunov exponent

\[ |s_1(t + \tau) - s_2(t + \tau)| = |s_1(t) - s_2(t)| e^{\lambda_1 \tau} \]

Solve for \( \lambda_1 \) and implement the limits.

Let \( s_1 \) and \( s_2 \) denote two near trajectory segments of the dynamics. The first Lyapunov exponent is defined as:

\[
\lambda_1 := \lim_{\tau \to \infty} \lim_{|s_1(t) - s_2(t)| \to 0} \frac{1}{\tau} \ln \left( \frac{|s_1(t+\tau) - s_2(t+\tau)|}{|s_1(t) - s_2(t)|} \right)
\]

Also: largest Lyapunov exponent or just Lyapunov exponent.
Further Lyapunov exponents

A perturbation aligns itself along the **direction** of strongest expansion / weakest contraction.

- this takes some time (A and B)
- direction depends on the current state (C)
- orthogonal directions for further Lyapunov exponents (D)
Lyapunov spectrum

Second Lyapunov exponent
The largest Lyapunov exponent determined with perturbations orthogonal to the direction corresponding to the first Lyapunov exponent

Third Lyapunov exponent
The largest Lyapunov exponent determined with perturbations orthogonal to the directions corresponding to the first two Lyapunov exponents

...  

- practical: frequent orthogonalizations  
- as many Lyapunov exponents as phase-space dimensions:

\[ \sum_{i=1}^{d} \lambda_i = \nabla \cdot f \]
# Lyapunov spectrum and type of Dynamics

For bounded, continuous-time dynamical systems, we have:

<table>
<thead>
<tr>
<th>Signs of Lyapunov Exponents</th>
<th>Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>-, --, ---, ...</td>
<td>fixed point</td>
</tr>
<tr>
<td>+, ++, ++++, ..., +0, ++0, ...</td>
<td>not possible (unbounded)</td>
</tr>
<tr>
<td>0, 00, 000, ...</td>
<td>no dynamics ($f = 0$)</td>
</tr>
<tr>
<td>0−, 0--, 0---, ...</td>
<td>periodic / limit cycle</td>
</tr>
<tr>
<td>00−, 00--, 00---, ...</td>
<td>quasiperiodic (torus)</td>
</tr>
<tr>
<td>000−, 0000−, ..., 000--−, ...</td>
<td>quasiperiodic (hypertorus)</td>
</tr>
<tr>
<td>+0−, +0--, +0---, ...</td>
<td>chaos</td>
</tr>
<tr>
<td>++0−, +++0−, ..., ++0--−, ...</td>
<td>hyperchaos</td>
</tr>
<tr>
<td>∞, ...</td>
<td>noise</td>
</tr>
</tbody>
</table>
Deterministic Chaos

No commonly accepted definition.

For our purposes:

“A bounded, deterministic dynamics with a positive Lyapunov exponent.”

The exponential divergence or convergence of nearby trajectories (Lyapunov exponents) is conceptually the most basic indicator of deterministic chaos.

M. Sano and Y. Sawada

Properties of Deterministic Chaos

**Necessary conditions:**
- in continuous-time dynamical systems: three dimensions or more
- non-linearity

**Properties:**
- sensitivity to initial conditions (butterfly effect) → only predictable on a short time scale
- no regularity
- fractal or strange attractors
- many more (→ future lectures)
So far:

- dynamical equations of motion known and fixed
- almost arbitrarily long time series (through simulation)
- high (unlimited) precision
- access to all dynamical variables
- no noise

Typical experimental situation:

- dynamical equations of motion unknown with changing parameters
- short time series
- low (limited) precision
- access to few (or only one) dynamical variables
- noise and uncertainties