Phase Space

Reconstruction
Brief Recap: Need for Nonlinear Methods

Lorenz oscillator

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= x(28 - z) - y \\
\dot{z} &= xy - \frac{8}{3}z
\end{align*}
\]

AR(1) process measured with nonlinearity

\[
\begin{align*}
y_t &= 0.8y_{t-1} + \epsilon_t \\
x_t &= (1 + y_t)^2
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from: G. Ansmann
Brief Recap: Need for Nonlinear Methods

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skewness ≈ 0.004; kurtosis ≈ -0.71

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skewness ≈ 2.6; kurtosis ≈ 9.7

indication for nonlinearity?

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Brief Recap: Need for Nonlinear Methods

When faced with time series from nonlinear systems, linear methods
- fail to detect the dynamics / structure in the data
- do not tell much about the dynamics
- cannot distinguish chaos from noise

→ Structure can be seen in attractors.
Brief Recap: Attractor

states of the dynamics for $t \to \infty$

type of dynamics can be deduced from topology of attractor:

- point $\to$ fixed-point dynamics
- limit cycle $\to$ periodic dynamics
- torus $\to$ quasiperiodic dynamics
- strange attractor $\to$ chaos

attractor reflects further central properties of dynamics.
Strange Attractors

Fundamentals of Analyzing Biomedical Signals

Phase Space
Need for Phase-Space Reconstruction

Directly observing the phase space / attractor requires access to all the system’s dynamical variables

But:
- often, only one dynamical variable accessible (or a time series thereof)
- dimension of phase space is often unknown

Can we obtain from a single time series a set that preserves important properties of the attractor?
Phase-Space Reconstruction

- **Actual attractor**
- **Perfect reconstruction**
- **Perfect reconstruction**
- **Preserves structure, practically bad**

- **Preserves structure, practically bad**
- **2D: structure not preserved**
- **3D: structure may be preserved**
- **(Phase as real number) structure not preserved**
- **(Forth and back on the same line) structure not preserved**

*from: G. Ansmann*
Phase-Space Reconstruction

original attractor
→ a $d$-manifold $A \subset \mathbb{R}^d$

measurement and reconstruction
→ a map $\phi : A \rightarrow \mathbb{R}^m$

structure-preserving reconstruction
→ topology-preserving map → an embedding

embedding
a map $\phi : A \rightarrow \mathbb{R}^m$ is called an embedding, if:
- $\nabla \phi$ has full rank
- $\phi$ is a diffeomorphism:
  - $\phi$ is differentiable
  - $\phi^{-1}$ exists and is differentiable
Phase-Space Reconstruction

**Strong Whitney embedding theorem**

For $m = 2d$, there exists a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ that is an embedding.

Problem: $\phi$ usually unknown

**Weak Whitney embedding theorem**

For $m > 2d+1$, almost every continuously differentiable ($C^1$) map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ is an embedding.

Problems:

- Often, we do not have $m$ independent observables
  (redundant observables are one of the reasons for “almost every”)
- We do not know $d$
Phase-Space Reconstruction

**Idea:**

- given time series $v: v_1, v_2, \ldots, v_N$ of some system observable $x$
- derivatives (first, second, third, ...) are not fully redundant.
- approximate derivatives with difference quotients:
  
  $\dot{v}_i = v_{i+1} - v_i$
  
  $\ddot{v}_i = v_{i+2} - 2v_{i+1} + v_i$
  
  etc.

$\sim v_i, v_{i+1}, \ldots$ are not fully redundant

$\sim$ inverse Taylor expansion
Phase-Space Reconstruction

**Takens’ Theorem:**

- let $\mathcal{A} = \{x_1, x_2, \ldots, x_N\}$ with the index indicating time
- let $h : \mathcal{A} \to \mathbb{R}$ denote the measurement function that maps the system observable $x$ to the time series $v$

If $m > 2d+1$,

$$\phi_{h,\tau} := (v_i, v_{i-\tau}, \ldots, v_{i-(m-1)\tau})$$

is an embedding for almost all dynamics, *embedding delays* $\tau$ and measurement functions $h$. $m$ denotes the *embedding dimension*. 
Phase-Space Reconstruction

**Takens’ Theorem and applications:**

- given time series $v$: $v_1, v_2, ..., v_N$ of some system observable $x$
- consider $m$-dimensional states (mapped from the attractor to the time series:

$$\begin{pmatrix} v_i, v_{i-\tau}, v_{i-2\tau}, \cdots, v_{i-(m-1)\tau} \end{pmatrix}^T$$

- for a proper embedding dimension $m$ and embedding delay $\tau$, these states make up a topologically equivalent reconstruction of the attractor.
Phase-Space Reconstruction

Example: Lorenz attractor

from: G. Ansmann
Phase-Space Reconstruction

Example: brain dynamics

EEG (awake state)
Phase-Space Reconstruction

Example: brain dynamics

EEG (epilepsy patient)
Phase-Space Reconstruction

Example: brain dynamics

EEG (epileptic seizure)
Dynamical Invariants

Important characteristics of the dynamics are invariant under the embedding transformation:

• Lyapunov exponents
• dimensions
• entropy
• ...

Phase-Space Reconstruction Delay-Embeddings
Phase-Space Reconstruction

Identifying embedding parameter

example: Lorenz attractor
Phase-Space Reconstruction

Identifying embedding parameter
delay

example: Lorenz attractor

from: G. Ansmann
Fundamentals of Analyzing Biomedical Signals

Phase Space

Phase-Space Reconstruction

Identifying embedding parameter
delay
example: Lorenz attractor

delay

from: G. Ansmann
Phase-Space Reconstruction

Identifying embedding parameter

delay

example: Lorenz attractor

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Phase-Space Reconstruction

Identifying embedding parameter
delay
example: Lorenz attractor

t = 0.20

t = 0.30

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Phase-Space Reconstruction

**Identifying embedding parameter**

requirement for an embedding:
\( \{v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau}\} \) not fully redundant

→ aforementioned theorems: almost every \( \tau \) yields an embedding:

requirements for a **good embedding**:

- minimum redundancy of \( \{v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau}\} \) (to unfold the attractor)
- reasonably small \( \tau \) (to avoid folding the attractor onto itself)

(compare to: linear independence vs. orthogonality)
Phase-Space Reconstruction

Identifying embedding parameter
using zeros of the autocorrelation

Idea:

if autocorrelation = 0 for some delay \( \Delta \) \( \Rightarrow \)
\( v_t \) and \( v_{t+\Delta} \) are \textit{linearly} independent on average

\( \rightarrow \) choose the first zero of the autocorrelation \( \Delta \) as embedding delay
Phase-Space Reconstruction

Identifying embedding parameter

using the first minimum of mutual information $I$

Idea: if common information for some delay $\Delta$ is minimum $\Rightarrow$ $v_t$ and $v_{t-\Delta}$ are independent on average (also includes nonlinear relationships)

$$I(M_1, M_2) = H(M_1) - H(M_1 | M_2) = H(M_1) + H(M_2) - H(M_1, M_2)$$

where $M_1$ and $M_2$ denote measurements at times $t$ and $t-\Delta$, and

$H = - \sum_i p_i \log p_i$ is the Shannon entropy

$\rightarrow$ choose the first minimum of the mutual information $\Delta$ as embedding delay

Identifying embedding parameter

first minimum of mutual information $I$

Phase-Space Reconstruction

Identifying embedding parameter
- zeros of the autocorrelation function
- minima of the mutual information
- many more

No method is perfect or commonly agreed upon.

Practically:
- try at least two methods
- judge by further analysis
- alternative for \( m \leq 3 \): visually inspect the attractor
- keep embedding window \((m - 1)\tau\) (time span in an embedded vector) constant
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Phase-Space Reconstruction

Identifying embedding parameter

dimension

embedding theorems define “sufficient” embedding dimension $m$

problem: dimension of system under study usually unknown

choosing $m$ overly high may hamper further analyses
(impact of noise, finite number of data points, computational complexity)

→ Need other ways to determine a good $m$
**Phase-Space Reconstruction**

*Identifying embedding parameter*

**Linear Dependence**

*idea:*
- $m$ is higher than necessary
  - attractor only covers a subspace of reconstruction space
    - (e.g., circle in $m = 3$)
- check whether embedded vectors have full rank.

*difficulties:*
- noise acts in all directions
- assumes linear dependence $\rightarrow$ dependence may be nonlinear
Phase-Space Reconstruction

**Identifying embedding parameter dimension**

### Asymptotic Invariants

**idea:**
- $m$ too small $\Rightarrow$ wrong dynamical invariants (in general)
- $m$ sufficient $\Rightarrow$ correct dynamical invariants

$\rightarrow$ increase $m$ until dynamical invariants converge

**difficulties:**
- criterion for convergence under real conditions (noise, finite number of data points, …)
- wrong (in general) is unpredictable
Phase-Space Reconstruction

**Identifying embedding parameter**

**False Nearest Neighbors**

idea:

- \( m \) too small \( \Rightarrow \) trajectories intersect
  \( \Rightarrow \) points close in reconstruction space that aren’t close in actual phase space (false nearest neighbors)

\( \rightarrow \) increase \( m \) until false nearest neighbors vanish.

Phase-Space Reconstruction

**Identifying embedding parameter**

**False Nearest Neighbors**

practically:
- choose threshold $\varepsilon$ for nearest neighbors
- $NN(m)$: number of pairs of points in $m$-dimensional reconstruction space that are closer than $\varepsilon$
- $NN(m + 1) < NN(m)$

$\Rightarrow$ at least $NN(m) - NN(m + 1)$ false nearest neighbors in the $m$-dimensional reconstruction.

difficulties:
number of true nearest neighbors large and fluctuating (noise).
Phase-Space Reconstruction

Summary

delay embedding allows to reconstruct attractor from single observable
- parameters $m$ and $\tau$ have to be carefully chosen
- reconstructed phase space may be used for:
  - understanding
  - prediction
  - modelling
  - …

- characteristics preserved by reconstruction:
  dimensions, Lyapunov exponents, entropy, …
Phase-Space Reconstruction

**Summary**

delay embedding allows to reconstruct attractor from single observable

- parameters $m$ and $\tau$ have to be carefully chosen
- reconstructed phase space may be used for:
  - understanding
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  - …

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  Lyapunov exponents, dimensions, entropy, …
Phase-Space Reconstruction

Extensions

• multivariate time series
• different embedding delays for each component
• state-dependent embedding delays