Lyapunov Exponents

Stability and Predictability from Time Series
Long-term Behavior and Stability

long-term:
\[ t \to \infty \] (can not be achieved when observing real systems)

observation time:
\[ T \ll \infty \]

largest characteristic time scale of system
\[ t_c < T \]
Long-term Behavior and Stability

different types of long-term behavior:

- *unlimited growth*
  in practice: can usually not be observed
  in model studies: temporal stabilization or change of model

- *bounded dynamics*
  fixed point, equilibrium
  periodic, quasi-periodic motion
  chaotic motion

Q: how stable is the dynamics,
when perturbing the system?
(when changing control parameters? mostly not considered)
Long-term Behavior and Stability

stability dogma (Andronov & Pontryagin, 1930s):

“since all mathematical models are simplifications and abstractions, models that are relevant for applications must be structurally stable”

however

simple models that are composed of physically acceptable unit are structurally unstable

(cf. weak/strong causality)

which (initial) states lead to the same / a similar long-term behavior?

→ concept of Lyapunov-stability
Long-term Behavior and Stability

Consider an, in general, nonlinear dynamical system

\[ \dot{x} = f(x(t), \beta), \quad x \in \mathbb{R}^d \]

Suppose \( f \) has an equilibrium at \( x_e \) so that \( f(x_e) = 0 \), then this equilibrium

- is Lyapunov stable, if for any \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \), such that if \( \|x(t = 0) - x_e\| < \delta \) then \( \|x(t) - x_e\| < \epsilon \) for every \( t \geq 0 \),

- is asymptotically stable, if it is Lyapunov stable and there exists \( \delta > 0 \) such that if \( \|x(t = 0) - x_e\| < \delta \), then \( \lim_{t \to \infty} \|x(t) - x_e\| = 0 \),

- is exponentially stable, if it is asymptotically stable and there exist \( \alpha > 0, \gamma > 0, \delta > 0 \) such that if \( \|x(t = 0) - x_e\| < \delta \), then \( \|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\gamma t} \), for all \( t \geq 0 \),

where \( \|\cdot\| \) denotes, e.g., the Euclidean or the Manhattan norm.

Fundamentals of Analyzing Biomedical Signals  

**Lyapunov Exponents**

**Long-term Behavior and Stability**

The aforementioned notions of *equilibrium stability* can be generalized to *orbital stability* (closed trajectory; i.e., periodic, quasi-periodic, or non-periodic orbit):

A trajectory $\Phi(t)$ is called *Lyapunov-stable* if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$, such that the trajectory of any solution $x(t)$ starting at the $\delta$-neighborhood of $\Phi(t)$ remains in the $\epsilon$-neighborhood of $\Phi(t)$ for all $t \geq 0$.

Linear stability analysis:

- consider small perturbation (of equilibrium/trajectory)
- expand $f$ in Taylor-series
- check eigenvalues of Jacobian; stable, if all have strictly negative parts
- real part of the largest eigenvalue (*Lyapunov exponent*) determines time to return to equilibrium/trajectory after perturbation
Long-term Behavior and Stability

*chaotic motion is (locally) Lyapunov-unstable:*

**divergence:**
distance between initially close trajectory segments grows exponentially in time (stretching mechanism)

**convergence:**
divergence of initially close segments limited by system size; when reached, distance shrinks again (folding mechanism)
Long-term Behavior and Stability

Example: two identical Lorenz oscillators with initial conditions; one oscillator is slightly perturbed \((10^{-14})\) at \(t = 30\)

**Graph:**
- \(x_1^2\) and \(x_1^1\) versus \(t\)
- Distance grows exponentially
- Growth limited by system size

(from: G. Ansmann)
Long-term Behavior and Stability

Example: logistic map:  \( x_{n+1} = r x_n (1 - x_n); \quad x_n \in [0, 1]; \quad r \in [0, 4] \)

\( \lambda_1 = \text{largest Lyapunov exponent} \)

(dominate the dynamics)
Fundamentals of Analyzing Biomedical Signals

Lyapunov Exponents

Long-term Behavior and Stability

spectrum of Lyapunov exponents: \( \lambda_i \) with \( i = 1, \ldots, d \)
characterize growth rates in different local directions of phase space

Lyapunov exponents and divergence:
\[
\sum_{i=1}^{d} \lambda_i = \nabla \cdot f
\]

dissipative system:
\[
\sum_{i=1}^{d} \lambda_i < 0
\]

largest Lyapunov exponent:
\[
\lambda_1 = 0 \rightarrow \text{regular dynamics}
\]
\[
\lambda_1 > 0 \rightarrow \text{chaotic dynamics}
\]
\[
\lambda_1 < 0 \rightarrow \text{fixed-point dynamics}
\]
\[
\lambda_1 \rightarrow \infty \rightarrow \text{stochastic dynamics}
\]
Lyapunov exponents from time series

**model:**
- continuous trajectories
- actual phase space
- evolution of arbitrary states
- equations of motion

**field data:**
- → discrete trajectories
- → reconstruction
- → available trajectories

and of course: finite data, noise, ...

**concepts and algorithms** (most widely used):
- spectrum of Lyapunov exponents (in general, hard to estimate)
  - (Sano & Sawada, 1985; Eckmann et al., 1986; Stoop & Parisi, 1991)
- largest Lyapunov exponent
  - (Wolf et al. 1985; Rosenstein et al., 1993; Kantz, 1994)
**Largest Lyapunov exponent**

Consider evolution of two nearby trajectory segments $s_1$ and $s_2$

For infinitesimally close trajectory segments ($\|s_1(t) - s_2(t)\| \to 0$) and for infinite time evolution ($\tau_e \to \infty$) the distance between segments grows or shrinks exponentially:

$$\|s_1(t + \tau_e) - s_2(t + \tau_e)\| = \|s_1(t) - s_2(t)\| e^{\lambda_1 \tau_e}$$
Largest Lyapunov exponent

\[ \| s_1(t + \tau_e) - s_2(t + \tau_e) \| = \| s_1(t) - s_2(t) \| e^{\lambda_1 \tau_e} \]

Solve for \( \lambda_1 \) and implement the limits.

Let \( s_1 \) and \( s_2 \) denote two near trajectory segments of the dynamics. The first Lyapunov exponent is defined as:

\[ \lambda_1 := \lim_{\tau_e \to \infty} \lim_{\| s_1(t) - s_2(t) \| \to 0} \frac{1}{\tau_e} \ln \left( \frac{\| s_1(t + \tau_e) - s_2(t + \tau_e) \|}{\| s_1(t) - s_2(t) \|} \right) \]

Also: largest Lyapunov exponent or just Lyapunov exponent.
Largest Lyapunov exponent estimates the dominant Lyapunov exponent from a time series by monitoring orbital divergence.

1. reconstruct phase space
2. pick $x(t_0)$ on *fiduciary* trajectory
3. find nearest neighbor $z_0(t_0)$
4. compute $||z_0(t_0) - x(t_0)|| =: L_0$
5. follow *difference* trajectory (dashed line forwards in time and compute $||z_0(t_i) - x(t_i)|| =: L_0(i)$. Increment $i$ until $L_0(i) > \epsilon$, call that value $L'_0$ and that time $t_1$
6. find $z_1(t_1)$, the *nearest neighbor* of $x(t_1)$, and loop to step 4. Repeat procedure to the end of *fiduciary* trajectory ($t = t_n$). Keep track of the $L_i$ and $L'_i$

Find largest (positive) Lyapunov exponent from:

$$\lambda_1 \approx \frac{1}{N \Delta t} \sum_{i}^{M-1} \log_2 \left( \frac{L'_i}{L_i} \right)$$

where $M$ denotes number of loops, and $N$ number of time steps on fiduciary trajectory; $N \Delta t = t_n - t_0$

Largest Lyapunov exponent

Limitations

- too many parameters that have to be chosen a priori
- problems may be obfuscated:
  - no exponential growth due to noise
  - embedding dimension $m$ too small
- highly sensitive to noise
- difficult to find neighboring trajectory segment with required properties

→ need a different way to ensure alignment to direction of largest growth
Largest Lyapunov exponent

1. choose reference state \( x(t_0) \)
   and all states \( x(t_1), \ldots, x(t_u) \)
in \( \varepsilon \)-neighborhood

2. for given \( \tau_e \), define average distance of respective trajectory
   segments from the initial one as
   \[
   s(t, \tau_e) := \frac{1}{u} \sum_{j=1}^{u} \| x(t_0 + \tau_e) - x(t_j + \tau_e) \|
   \]

3. average over all states as reference states:
   \[
   S(\tau_e) := \frac{1}{N} \sum_{t=1}^{N} s(t, \tau_e)
   \]

4. obtain largest Lyapunov exponent from region of exponential
growth of \( S(\tau_e) \)
Largest Lyapunov exponent

Rosenstein-Kantz Algorithm

Fig. 2. $S(\tau)$ for a Hénon trajectory of length 2000. The different curves correspond to $\epsilon = 0.0005$, 0.002 and 0.008 (the three bunches from bottom to top) and embedding dimension $m = 2-5$. The dashed lines have slopes $\lambda_{\text{exact}} = 0.4169$. Again $\tau \leq 0$ corresponds to the components used to define the local neighbourhoods.
Largest Lyapunov exponent

Mind how you average
1. average over the neighborhood of a reference state $\rightarrow s(t, \tau_e)$
2. average $s(t, \tau_e)$ over all reference states $\rightarrow S(\tau_e)$
3. Obtain $\lambda_1$ from slope of $S(\tau_e)$

Density of states in a region of the attractor affects:
- reference states
- states in neighborhood of given reference state

Separating averaging in steps 1 and 2 (instead of averaging of all $\epsilon$-close pairs) ensures that density is accounted for only once (and not twice)
Advantages and Problems

- region of exponential growth can be determined a posteriori (be careful of wishful thinking though)

- absence of exponential growth usually detectable (but only usually)

- region of strong noise influence can be detected and excluded

- can only determine the largest Lyapunov exponent
Largest Lyapunov exponent

• tangent-space methods
  → require estimate of Jacobian

• spectrum of Lyapunov exponents
  → requires a lot of data
Largest Lyapunov exponent

for flows: inverse seconds

for maps: inverse iterations

other choices: bits/second or bits/iteration
**Spectrum of Lyapunov exponents and type of dynamics**

For bounded, continuous-time dynamical systems, we have:

<table>
<thead>
<tr>
<th>signs of Lyapunov exponents</th>
<th>dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>−, −−, −−−, ...</td>
<td>fixed point</td>
</tr>
<tr>
<td>+, ++, ++++, ..., +0, ++0, ...</td>
<td>not possible (unbounded)</td>
</tr>
<tr>
<td>0, 00, 000, ...</td>
<td>no dynamics ($f = 0$)</td>
</tr>
<tr>
<td>0−, 0---, 0----, ...</td>
<td>periodic / limit cycle</td>
</tr>
<tr>
<td>00−, 00--, 00---, ...</td>
<td>quasiperiodic (torus)</td>
</tr>
<tr>
<td>000−, 0000−, ..., 000--−, ...</td>
<td>quasiperiodic (hypertorus)</td>
</tr>
<tr>
<td>+0−, +0--, +0---, ...</td>
<td>chaos</td>
</tr>
<tr>
<td>++0−, +++0−, ..., +0--−, ...</td>
<td>hyperchaos</td>
</tr>
<tr>
<td>∞, ...</td>
<td>noise</td>
</tr>
</tbody>
</table>
Largest Lyapunov exponent

field applications

- number of data points \( \lim N \to \infty \)
- data precision
  adopt to requirement of small \( \varepsilon \)-neighborhood
- strong correlations in data (sampling interval)
  use Theiler correction (see Dimensions)
- noise
  similar impact as with Dimensions
- filtering
  classical filter affect negative Lyapunov exponents only
due to adding a (passive) system \( \Rightarrow \) extra Lyapunov exponent
magnitude \( \sim \) cutoff frequency
Largest Lyapunov exponent

False indications of chaos:

- unbounded orbits can have $\lambda_1 > 0$

- orbits can separate but not exponentially

  (check boundedness and be sure orbit has adequately sampled attractor; check for contraction to zero within machine precision)

- can have transient chaos*

  (double-check with other methods)

*e.g., Y.-C. Lai and T. Tél, Transient Chaos, Complex Dynamics on Finite-Time Scales (Springer, New York, 2011)
Largest Lyapunov exponent

transient chaos: an example
Largest Lyapunov exponent

- stability and type of the dynamics:
  \( \lambda_1 > 0 \) chaos, unstable dynamics
  \( \lambda_1 = 0 \) regular dynamics
  \( \lambda_1 < 0 \) fixed-point dynamics

- quantification of loss of information due to action of nonlinearity

- prediction horizon:

\[
T_p \approx \frac{-\ln(\rho)}{\sum_{i, \lambda_i > 0} \lambda_i}
\]

where:
- \( \rho \) denotes accuracy of measurement (initial state)
- \( \sum_{i, \lambda_i > 0} \) is sum of positive Lyapunov exponents
Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents

\[ D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \text{ where } \sum_{i=1}^{k} \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k+1} \lambda_i < 0 \]

Kaplan-Yorke dimension \( D_{KY} \) equals information dimension \( D_1 \)
(Note: conjecture not generally valid!)

example:
- Hénon map with parameters \( a = 1.4 \) and \( b = 0.3 \)
- \( \lambda_1 = 0.603, \lambda_2 = -2.34 \)
- we find with \( k = 1 \):

\[ D_{KY} = k + \frac{\lambda_1}{|\lambda_2|} = 1 + \frac{0.603}{|-2.34|} = 1.26 \]
Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents