Fractals, Dimensions, and Fractal Dimensions

Gerrit Ansmann
Sierpiński Carpet
Koch Snowflake
Generating Fractals
Generating Fractals
Generating Fractals
Generating Fractals

[Diagram showing the generation of a fractal through recursive division and multiplication.]
Generating Fractals

[Diagram showing the generation of a fractal through iterative processes.]
Generating Fractals
Generating Fractals

![Fractal images](image.png)
Generating Fractals

Sierpiński Triangle
No commonly accepted definition (arguably none is needed). Nonetheless:

**Fractal**

A set $\mathcal{F}$ with:

- self-similarity: a subset of $\mathcal{F}$ is similar to $\mathcal{F}$
- fine structure

**Applications:**

- many natural structures
- modelling
- technology
- art
- many more
blood vessels in the mouse brain
Q: What does the “B.” in “Benoît B. Mandelbrot” stand for?
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A: “Benoît B. Mandelbrot”

possibly not a joke
Strange Attractors are Like Fractals

Logistic map:

\[ x_{t+1} = f(x_t) = r x_t (1 - x_t) \quad x_0 \in [0; 1] \]
Attractor of the Logistic Map

![Diagram of the attractor of the Logistic Map](image)

PDF

$x_t$

Dimensions: 362.8x272.1
Strange Attractors are Like Fractals

Attention: Handwaving

Logistic map:

\[ x_{t+1} = f(x_t) = rx_t(1 - x_t) \quad x_0 \in [0; 1] \]

- Consider strange attractor \( \mathcal{A} \)
- \( \mathcal{A} \) forward-invariant:
  \[ f(\mathcal{A} \cap [0, \frac{1}{2}]) \subset \mathcal{A} \]
- \( \mathcal{A} \) forward-invariant:
  \[ f(\mathcal{A} \cap [\frac{1}{2}, 1]) \subset \mathcal{A} \]
\( \Rightarrow \) self-similarity

Problem: “\( \subset \)”, not “\( = \)"
Strange Attractors are Like Fractals

Analogon for a continuous-time dynamical systems:
Attractor of the Lorenz System

adapted from commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png
Fractal Parameter Regimes (Shrimps)
Fractal Attractor Basin (Duffing Oscillator)

Two-dimensional slice of three-dimensional parameter space.

cyan / red: projection of attractors.
What is the dimension of a fractal or strange attractor?
Classical Dimensions

Some classic dimensions:

- vector-space dimension
- manifold dimension

→ number of real numbers needed to identify a point of the object (with some algebraic structure preserved)

Problems:

- dimension of state space unknown, embedding dimension arbitrary
- strange attractors are no manifolds
Topological Dimensions

Idea for the inductive dimension:

• The boundary of a cube consists of squares.
• The boundary of a square consists of lines.
• The boundary of a line consists of points.
• A point has dimension 1.

Inductive dimension (sketch)

• $d_i(\emptyset) := -1$
• $d_i(S) := 1 + \max_{\mathcal{V} \subseteq S} d_i(\partial \mathcal{V})$, where the $\mathcal{V}$ are open neighbourhoods.
Topological Dimensions

Inductive dimension (sketch)

- $d_i(\{\}) := -1$
- $d_i(S) := 1 + \max_{\mathcal{V} \subset S} d_i(\partial \mathcal{V})$, where the $\mathcal{V}$ are open neighbourhoods.

Lebesgue covering dimension (sketch)

Consider fine covers of $S$ with subsets. Then $d_L(S) + 1$ is the number of unavoidable overlaps of subsets in a cover.
Topological Dimensions

Inductive dimension (sketch)

- \( d_i(\{\}) := -1 \)
- \( d_i(\mathcal{S}) := 1 + \max_{\mathcal{V} \subset \mathcal{S}} d_i(\partial \mathcal{V}) \),
  where the \( \mathcal{V} \) are open neighbourhoods.

Lebesgue covering dimension (sketch)

Consider fine covers of \( \mathcal{S} \) with subsets. Then \( d_L(\mathcal{S}) + 1 \) is the number of unavoidable overlaps of subsets in a cover.

Problems:

- different for similar objects
- same for dissimilar objects
**Similar Construction; Different Topological Dimension**

![Diagram showing the similar construction with different topological dimension. The initial shape is a square, and after a transformation, it becomes a more complex structure with additional 'F's arranged in a specific pattern.](image-url)
Different Construction; Same Topological Dimensions

(Next slides.)
Generalised Dimensions

Idea:

If you multiply all lengths by $a$,

- lengths will change by a factor $a$
- areas will change by a factor $a^2$
- volumes will change by a factor $a^3$
- ...

→ determine dimension by exponent of content-scaling.
The Box-Counting Dimension

Ideas:
- Determine content via boxes covered by object.
- Don’t scale the object, but the boxes.

Box-counting dimension / Minkowski–Bouligand dimension

Let \( n(\mathcal{S}, \varepsilon) \) be the number of boxes of an \( \varepsilon \)-grid that contain at least one element of \( \mathcal{S} \).

\[
d_0(\mathcal{S}) := - \lim_{\varepsilon \to 0} \frac{\ln(n(\mathcal{S}, \varepsilon))}{\ln(\varepsilon)}
\]

Not the Hausdorff dimension (but yields the same result in most cases).
Box Counting – Example

\[ n(\mathcal{S}, \epsilon) = \begin{cases} 13 & \text{for } \epsilon = 13 \\ 26 & \text{for } \epsilon = 1/2 \\ 53 & \text{for } \epsilon = 1/4 \\ 102 & \text{for } \epsilon = 1/8 \end{cases} \]

\[ \dim_{\mathcal{S}} = -\lim_{\epsilon \to 0} \frac{\log n(\mathcal{S}, \epsilon)}{\log 1/\epsilon} = -\lim_{\epsilon \to 0} \frac{\log 13}{\log 1/\epsilon} = 1 \]
Box Counting – Example

- \( n(\mathcal{S}, 1) = 13 \)
Box Counting – Example

- $n(\mathcal{S}, 1) = 13$
- $n(\mathcal{S}, \frac{1}{2}) = 26$
Box Counting – Example

- $n(\mathcal{S}, 1) = 13$
- $n(\mathcal{S}, \frac{1}{2}) = 26$
- $n(\mathcal{S}, \frac{1}{4}) = 53$
Box Counting – Example

- \( n(\mathcal{S}, 1) = 13 \)
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Box Counting – Example

- $n(\mathcal{S}, 1) = 13$
- $n(\mathcal{S}, \frac{1}{2}) = 26$
- $n(\mathcal{S}, \frac{1}{4}) = 53$
- $n(\mathcal{S}, \frac{1}{8}) = 102$

$\rightarrow n(\mathcal{S}, \varepsilon) \approx \frac{13}{\varepsilon}$

$d_0(\mathcal{S}) = -\lim_{\varepsilon \to 0} \frac{\ln(n(\mathcal{S}, \varepsilon))}{\ln(\varepsilon)} = -\lim_{\varepsilon \to 0} \frac{\ln\left(\frac{13}{\varepsilon}\right)}{\ln(\varepsilon)} = -\lim_{\varepsilon \to 0} \frac{\ln(13)}{\ln(\varepsilon)} - 1 = 1$
Box Counting – Another Example

- $n(\mathcal{S}, 1) = 1$
- $n(\mathcal{S}, 1/3) = 5$
- $n(\mathcal{S}, 1/9) = 25$
- $n(\mathcal{S}, 1/27) = 125$

$\Rightarrow n(\mathcal{S}, \epsilon) \approx 5 \cdot \nu(\epsilon)$

$\delta_0(\mathcal{S}) = -\nu(\epsilon) \rightarrow \nu(\epsilon) = \lim_{k \to \infty} \frac{n(\mathcal{S}, \epsilon)}{k}$

$\nu(3) = \nu(5)$
Box Counting – Another Example

- \( n(\mathcal{S}, 1) = 1 \)
Box Counting – Another Example

- $n(S, 1) = 1$
- $n(S, \frac{1}{3}) = 5$
Box Counting – Another Example

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Box Counting – Another Example

\[ d_0(S) = - \lim_{\varepsilon \to 0} \frac{\ln(n(S, \varepsilon))}{\ln(\varepsilon)} = \lim_{k \to \infty} \frac{\ln(5^k)}{-\ln\left(\frac{1}{3^k}\right)} = \lim_{k \to \infty} \frac{k \ln(5)}{k \ln(3)} = \frac{\ln(5)}{\ln(3)} \]

- \( n(S, 1) = 1 \)
- \( n(S, \frac{1}{3}) = 5 \)
- \( n(S, \frac{1}{9}) = 25 \)
- \( n(S, \frac{1}{27}) = 125 \)

\[ \Rightarrow n(S, \varepsilon) \approx 5^{\log_3(\varepsilon)} \]
$d_B = 1.465$
Sierpiński Carpet
\[ d_B = 1.893 \]
$d_B = 1.262$
$d_B = 1.262$
Sierpiński Triangle

$\delta_B = 1.585$
Attractor of the Logistic Map
Rényi Dimension

Problem:
box counting ignores how densely the boxes are populated.

→ Weight boxes by probability $p_{\mathcal{S}\varepsilon}(i)$
to find element of $\mathcal{S}$ in box $i$.

Rényi dimension / $q$-dimension

$$d_q(\mathcal{S}) := \lim_{a\varepsilon \to 0} \frac{\ln \left( \sum_i p_{\mathcal{S}\varepsilon}^q (i) \right)}{(q - 1) \ln(\varepsilon)}$$
Rényi Dimension

- \(d_0 := \lim_{q \to 0} d_q\): box-counting dimension
- \(d_1 := \lim_{q \to 1} d_q\): information dimension
- \(d_2\): correlation dimension
- \(d_{\text{topological}} \leq d_{\text{Hausdorff}} \leq \ldots \leq d_2 \leq d_1 \leq d_0 \leq m\)
- in most cases: \(d_{\text{Hausdorff}} = \ldots = d_2 = d_1 = d_0\)

Fractal (alternative definition)

A set \(\mathcal{S}\) with \(d_{\text{topological}}(\mathcal{S}) < d_{\text{fractal}}(\mathcal{S})\).

- \(d_{\text{topological}} \in \{d_L, d_i\}\).
- \(d_{\text{fractal}} \in \{d_{\text{Hausdorff}}, d_0, d_1, \ldots\}\).
Kaplan–Yorke Dimension

Kaplan–Yorke dimension

Let $\hat{\Lambda}$ be the piece-wise linear interpolation of $\Lambda(k) := \sum_{i=1}^{k} \lambda_i$.

Then $d_{KY}$ is defined via $\hat{\Lambda}(d_{KY}) := 0$.

Kaplan–Yorke conjecture

$$d_{KY} = d_{fractal}$$

- possible sanity check
- needs many Lyapunov exponents
Motivation:

- $d \notin \mathbb{N}$: strong indicator for non-linearity
- $d > 1$: hint at non-linearity
- characterises self-similarity
- hints for modelling (degrees of freedom, attractor structure)
- sanity check via embedding theorems
Rényi Dimension – Practical Considerations

Rényi dimension

\[ d_q(\mathcal{S}) := \lim_{\varepsilon \to 0} \frac{\ln \left( \sum_i p_{\mathcal{S} \varepsilon}^q (i) \right)}{(q - 1) \ln(\varepsilon)} \]

- Replace limes by slope in double-logarithmic plot.
Rényi Dimension – Practical Considerations

Rényi dimension

\[ d_q(S) := \lim_{\varepsilon \to 0} \frac{\ln\left( \sum_i p_{S,\varepsilon}^q (i) \right)}{(q - 1) \ln(\varepsilon)} \]

- Replace limes by slope in double-logarithmic plot.
- Approximate: \( p_{S,\varepsilon} (i) \approx \frac{n(S,i)}{N} \).
Rényi Dimension – Practical Considerations

Rényi dimension

\[ d_q(\mathcal{S}) := \lim_{\varepsilon \to 0} \frac{\ln\left(\sum_i p_{\mathcal{S},\varepsilon}(i)\right)}{(q - 1) \ln(\varepsilon)} \]

- Replace limes by slope in double-logarithmic plot.
- Approximate: \( p_{\mathcal{S},\varepsilon}(i) \approx \frac{n(\mathcal{S},i)}{N} \).
- Approximate further: \( \sum_i p_{\mathcal{S},\varepsilon}(i) \approx C_q(\varepsilon) \).

Correlation sum

\[ C_q(\mathcal{S},\varepsilon) := \frac{1}{N} \sum_i \left( \frac{1}{N} \sum_j \Theta(\varepsilon - \|\vec{x}_i - \vec{x}_j\|) \right)^{q-1} \]

number of points closer than \( \varepsilon \)
Correlation Dimension

\[ d_2(\mathcal{S}) = \lim_{\varepsilon \to 0} \frac{\ln(\sum_i p_{\mathcal{S}\varepsilon}(i))}{\ln(\varepsilon)} \approx \lim_{\varepsilon \to 0} \frac{C_2(\mathcal{S}, \varepsilon)}{\ln(\varepsilon)} \]

- quickest to calculate

- \( C_2(\mathcal{S}, \varepsilon) \): fraction of pairs of points closer than \( \varepsilon \)
Example: Sine Wave
Example: Sine Wave

\[ d^2 \]

\[ \varepsilon / \text{diam } \mathcal{A} \]

\[ m = 1 \]
\[ m = 2 \]

scaling region
Example: Gaussian Noise

\[ C_2(\mathcal{A}, \varepsilon) \]

\[ \varepsilon / \text{diam } \mathcal{A} \]
Example: Gaussian Noise

\[ d_2 \]

\[ \varepsilon / \text{diam} \mathcal{A} \]
How many dimensional is a plate of spaghetti? Zero when seen from a long distance, two on the scale of the plate, one on the scale of the individual noodles and three inside a noodle. Maccaroni is even worse.

attributed to Peter Grassberger
**Example: Torus**

*simulating macaroni:*

Sum of two incommensurable sines (tube/torus) and some noise (dough):
**Torus: Correlation Sum**

The image shows a graph of $C_2(\mathcal{A}, \varepsilon)$ against $\varepsilon / \text{diam } \mathcal{A}$. The graph includes several curves for different values of $m$:

- $m = 1$
- $m = 2$
- $m = 3$
- $m = 4$
- $m = 5$
- $m = 6$

The x-axis represents $\varepsilon / \text{diam } \mathcal{A}$, and the y-axis represents $C_2(\mathcal{A}, \varepsilon)$ on a logarithmic scale.
Torus: Correlation Sum

\[ C_2(\mathcal{A}, \varepsilon) \]

\[ \varepsilon / \text{diam} \mathcal{A} \]

\[ m = 1 \]
\[ m = 2 \]
\[ m = 3 \]
\[ m = 4 \]
\[ m = 5 \]
\[ m = 6 \]
\[ \cdots \]
**Torus: Correlation Sum**

- $C_2(\mathcal{A}, \varepsilon)$
- $\varepsilon / \text{diam } \mathcal{A}$

- $m = 1$
- $m = 2$
- $m = 3$
- $m = 4$
- $m = 5$
- $m = 6$
- ...
Torus: Correlation Sum

$C_2(\mathcal{A}, \varepsilon)$

embedding dimension too small

$m = 1$
$m = 2$
$m = 3$
$m = 4$
$m = 5$
$m = 6$

$\varepsilon / \text{diam } \mathcal{A}$
Torus: Correlation Sum

\[ C_2(\mathcal{A}, \varepsilon) \]

embedding dimension too small

\[ m = 1 \quad m = 2 \quad m = 3 \quad m = 4 \quad m = 5 \quad m = 6 \]

\[ \varepsilon / \text{diam } \mathcal{A} \]
**Torus: Correlation Sum**

![Graph](image)

- **Legend:**
  - $m = 1$
  - $m = 2$
  - $m = 3$
  - $m = 4$
  - $m = 5$
  - $m = 6$
  - $\ldots$

- **Equations and Expressions:**
  - $C_2(\mathcal{A}, \varepsilon)$
  - $\varepsilon / \text{diam} \mathcal{A}$

- **Observations:**
  - Embedding dimension too small

- **Axes:**
  - Y-axis: $10^{-12}$ to $10^1$
  - X-axis: $0.0001$ to $10$

- **Additional Notes:**
  - $m = 1$ to $m = 6$ represent different embedding dimensions.
Torus: Dimension

\[
d^2 = \frac{\varepsilon}{\text{diam } \mathcal{A}}
\]

- \( m = 1 \)
- \( m = 2 \)
- \( m = 3 \)
- \( m = 4 \)
- \( m = 5 \)
- \( m = 6 \)
Theiler Correction

Problem:

- For a sufficiently fine temporal solution, points close in time are also close in phase space
  → correlation sum over-estimated

Theiler correction

Exclude temporally close points from the correlation sum:

$$\sum_{i,j} \Theta(\varepsilon - |\tilde{x}_i - \tilde{x}_j|) \rightarrow \sum_{i,j} \Theta(\varepsilon - |\tilde{x}_i - \tilde{x}_j|)$$

where

$$|i-j| > \vartheta$$

(adjust normalisation accordingly)

Can usually be applied without harm for reasonably small $\vartheta$. 
Lorenz System without Theiler Correction

\[ \begin{align*} d_2 & = \varepsilon / \text{diam } \mathcal{A} \\ m & = 1 \\ m & = 2 \\ m & = 3 \\ m & = 4 \\ m & = 5 \\ m & = 6 \\ m & = 7 \\ m & = 8 \end{align*} \]
Lorenz System with Theiler Correction

\[ d_2 = \frac{\epsilon}{\text{diam } A} \]

- \( m = 1 \)
- \( m = 2 \)
- \( m = 3 \)
- \( m = 4 \)
- \( m = 5 \)
- \( m = 6 \)
- \( m = 7 \)
- \( m = 8 \)
Actual solution $d_{\text{fractal}} \approx 2.05$

~ temporal evolution of $x$ and $y$ very similar
~ large regions of the attractor are a plane
Attractor of the Lorenz System

adapted from commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png
Hénon Attractor
Hénon Attractor
Hénon Attractor

\[ x_{t+1} = 1 - a x_t^2 + y_t, \]
\[ y_{t+1} = b x_t, \]

with parameters \( a = 1.4 \) and \( b = 0.3 \).
Hénon Attractor
Hénon Attractor
Hénon Attractor

\[ y_t \]

\[ x_t \]
Hénon Attractor
Dimension of the Hénon Attractor

\[ d_2 = \frac{\varepsilon}{\text{diam } \mathcal{A}} \]

- \( m = 1 \)
- \( m = 2 \)
- \( m = 3 \)
- \( m = 4 \)
- \( m = 5 \)
- \( m = 6 \)
- \( m = 7 \)
- \( m = 8 \)
Estimate dimension of time series via slope of correlation sum

- Compare multiple embedding dimensions \( m \).
- Select scaling region properly.
- Apply Theiler correction.

\[
\begin{align*}
    d_{\text{fractal}} \notin \mathbb{N} & : \text{chaotic dynamics} \\
    d_{\text{fractal}} \in \mathbb{N} & : \text{hint at regular dynamics} \\
    d_{\text{fractal}} = \infty & : \text{noise}
\end{align*}
\]