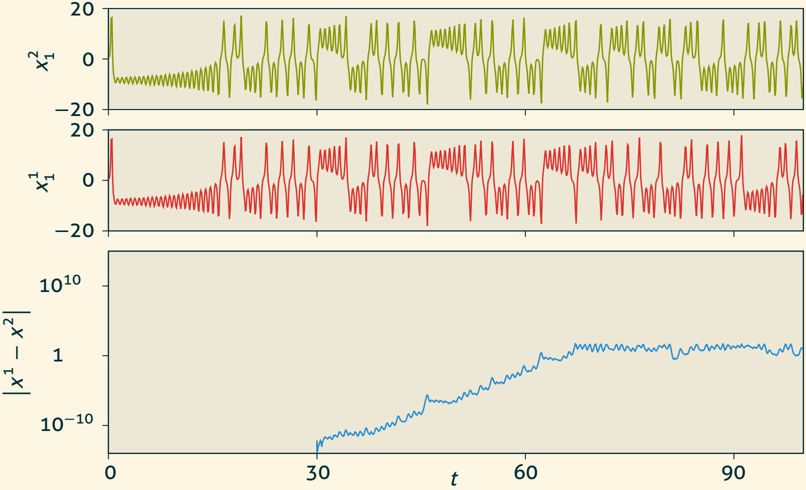
# Estimating Lyapunov Exponents from Time Series

Gerrit Ansmann

# Example: Lorenz system

Two identical Lorenz systems with initial conditions; one is slightly perturbed ($10^{-14}$) at $t = 30$:



# The Largest Lyapunov Exponent

Consider evolution of two close trajectories $x$ and $y$:


Then their distance grows or shrinks exponenentially:

$$|x(t+\tau) - y(t+\tau)| = |x(t) - y(t)|\, e^{\lambda_1 \tau}$$

For:

- infinitesimally close trajectories ($|x(t) - y(t)| \to 0$)
- infinite time evolution ($\tau \to \infty$)

*Note: In this entire lecture, $\tau$ is **not** the embedding delay.*

# The Largest Lyapunov Exponent – Definition

$$|x(t+\tau) - y(t+\tau)| = |x(t) - y(t)|\, e^{\lambda_1 \tau}$$

→ Solve for $\lambda_1$ and implement the limits:

**First Lyapunov exponent**

Let $x$ and $y$ be two trajectories of the dynamics.

$$\lambda_1 := \lim_{\tau \to \infty} \lim_{|x(t)-y(t)| \to 0} \frac{1}{\tau} \ln\left( \frac{|x(t+\tau) - y(t+\tau)|}{|x(t) - y(t)|} \right)$$

Also: *largest Lyapunov exponent* or just *Lyapunov exponent.*

# Typical Evolution of a Trajectory Distance

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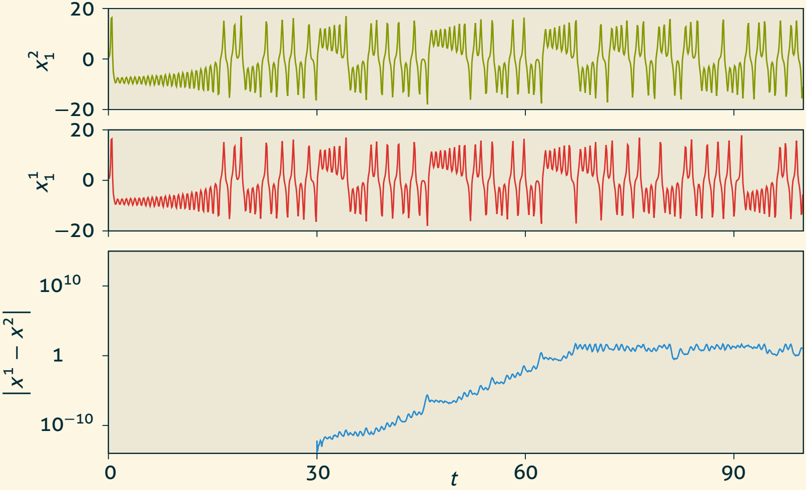
one is slightly perturbed ($10^{-14}$) at $t = 30$:



# Comparison with Evolution of Infinitesimal Distance

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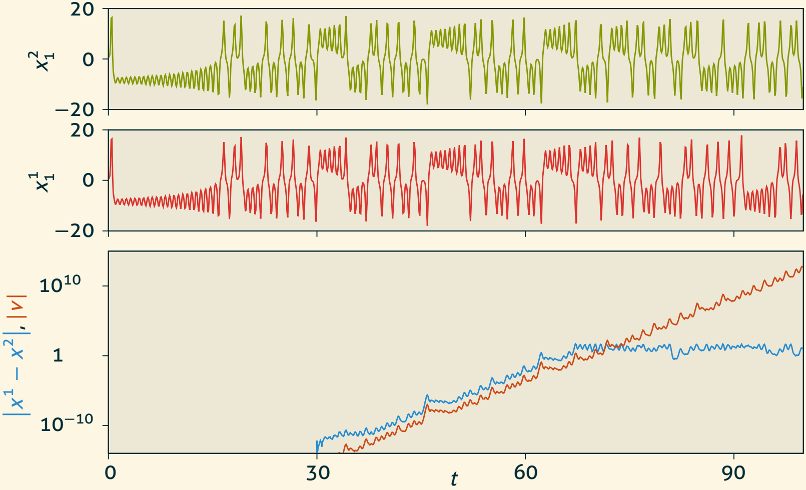
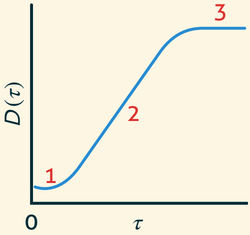


# Typical Evolution of a Trajectory Distance

Regimes of the average distance $D$:



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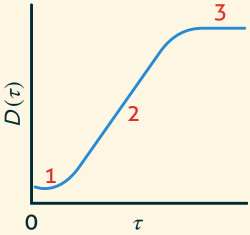


1. alignment to direction of largest growth:

$$D(\tau) \propto \sum_{i=1}^{d} c_i(\tau) \exp\left(\lambda_i \tau\right)$$

Asymptotically: $\frac{c_i(\tau)}{c_1(\tau)} \to 0$ for $i > 1$

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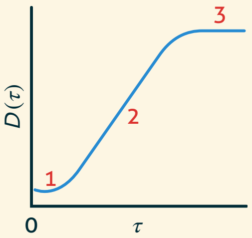


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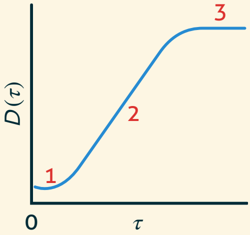


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2. exponential growth:

$$D(\tau) \propto \exp\left(\lambda_1 \tau\right)$$

3. constancy on the scale of the attractor:

$$D(\tau) \approx \mathrm{diam}(\mathscr{A})$$

# Translation to Time Series

- continuous trajectories
  → discrete trajectories

- actual phase space
  → reconstruction

- evolution of arbitrary states
  → available trajectories

And of course: finite data, noise, …

# Wolf Algorithm
# Acquisition of Instantaneous Lyapunov Exponents


1. Let $x(0)$ be the first reconstructed state.
2. Find a state $x(t_1)$ such that $\left|x(t_1) - x(0)\right| < \varepsilon$.

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3. Approximate instantaneous largest Lyapunov exponent:
   $\hat{\lambda}_1(0) = \frac{1}{\tau} \ln\left(\frac{x(\tau) - x(t_1+\tau)}{x(0) - x(t_1)}\right)$.

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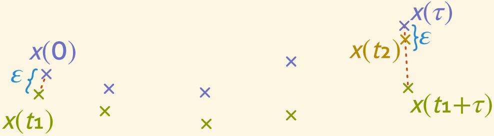


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4. Find a state $x(t_2)$ such that $\left|x(t_2) - x(\tau)\right| < \varepsilon$ and
   $x(t_2) - x(t + \tau)$ is nearly parallel to $x(t_1 + \tau) - x(t + \tau)$.

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5. Approximate instantaneous largest Lyapunov exponent:
   $\hat{\lambda}_1(\tau) = \frac{1}{\tau} \ln\left(\frac{x(2\tau)-x(t_2+\tau)}{x(\tau)-x(t_2)}\right)$.

# Wolf Algorithm – Averaging

After acquiring the local Lyapunov exponents, estimate:

$$\lambda_1 = \frac{1}{I - r} \sum_{i=r}^{I} \hat{\lambda}_1(i\tau)$$

The offset $r$ ensures that the distances are aligned to the direction of largest growth.

# Wolf Algorithm – Parameters


Initial distance $\varepsilon$:

- too small → impact of noise too high
- too large → small region of exponential growth

Rescaling time $\tau$:

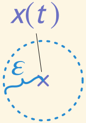
- too high → distance reaches size of attractor
- too small → small region of exponential growth

# Wolf Algorithm – Problems

- Parameters have to be chosen a priori.
- Problems may be obfuscated:
  - no exponential growth due to noise
  - embedding dimension $m$ too small
- Sensitivity to noise.
- Difficult to find neighbouring trajectory segment with required properties.

→ Different way to ensure alignment to direction of largest growth.

# Rosenstein–Kantz Algorithm




1. For a given *reference* state $x(t)$, find all states $x(t_1), \ldots, x(t_u)$ for which $\left|x(t) - x(t_j)\right| < \varepsilon$.

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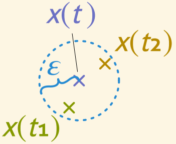


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$$s(t,\tau) := \frac{1}{u}\sum_{j=1}^{u}\left|x(t+\tau) - x(t_j+\tau)\right|$$

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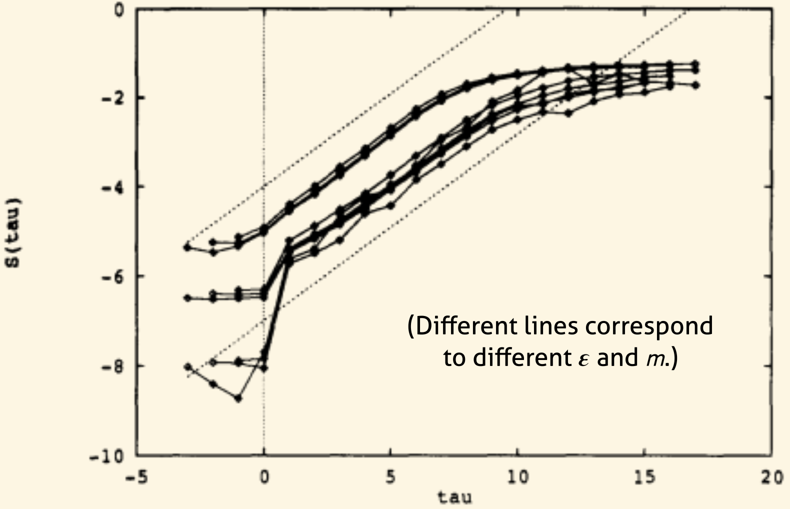
4. Obtain $\lambda_1$ from region of exponential growth of $S(\tau)$.

# Rosenstein–Kantz Algorithm – Example




adapted from H. Kantz, *A robust method to estimate the maximal Lyapunov exponent of a time series*, Phys. Let. A 185 (1994)

# Rosenstein–Kantz Algorithm – Mind How You Average

1. Average over the neighbourhood of a reference state $\rightarrow s(t,\tau)$.
2. Average $s(t,\tau)$ over all reference states $\rightarrow S(\tau)$.
3. Obtain $\lambda_1$ from slope of $S(\tau)$.

Density of states in a region of the attractor affects:

- reference states
- states in neighbourhood of given reference state

Separating the averaging in steps 1 and 2
(instead of averaging of all pairs closer than $\varepsilon$)
ensures that density is accounted for only once
(and not twice).

# Rosenstein–Kantz Algorithm – Advantages and Problems

- Region of exponential growth
  can be determined a posteriori.
  Be careful of wishful thinking though.

- Absense of exponential growth usually detectable
  (but only usually).

- Region of strong noise influence
  can be detected and excluded.

- Can only determine the largest Lyapunov exponent.

# Extensions and Alternatives

- tangent-space methods
  → require estimate of Jacobian

- further Lyapunov exponents
  → requires a lot of data

# Lyapunov Spectrum and Types of Dynamics

For bounded, continuous-time dynamical systems:

| signs of Lyapunov exponents | Dynamics |
|---|---|
| −, −−, −−−, … | fixed point |
| +, ++, +++, …, +0, ++0, … | not possible (unbounded) |
| 0, 00, 000, … | no dynamics ($f = 0$) |
| 0−, 0−−, 0−−−, … | periodic / limit cycle |
| 00−, 00−−, 00−−−, … | quasiperiodic (torus) |
| 000−, 0000−, …, 000−−, … | quasiperiodic (hypertorus) |
| +0−, +0−−, +0−−−, … | chaos |
| ++0−, +++0−, …, ++0−−, … | hyperchaos |
| ∞, … | noise |

# Interpretation

- Stability and type of the dynamics:

| | |
|---|---|
| $\lambda_1 > 0$ | chaos, instable dynamics |
| $\lambda_1 = 0$ | regular dynamics |
| $\lambda_1 < 0$ | fixed-point dynamics |

- Quantification of loss of information.
- Prediction horizon:

$$\tau_\mathrm{p} \approx \frac{-\ln(\rho)}{\sum_{i,\lambda_i>0} \lambda_i}$$

  - $\rho$: Accuracy of measurement (initial state).
  - $\sum_{i,\lambda_i>0}$: Sum of positive Lyapunov exponents.