Estimating Lyapunov Exponents from Time Series

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Example: Lorenz system

Two identical Lorenz systems with initial conditions; one is slightly perturbed ($10^{-14}$) at $t = 30$:
The Largest Lyapunov Exponent

Consider evolution of two close trajectories $x$ and $y$:

Then their distance grows or shrinks exponentially:

$$|x(t + \tau) - y(t + \tau)| = |x(t) - y(t)| e^{\lambda_1 \tau}$$

For:

- infinitesimally close trajectories ($|x(t) - y(t)| \to 0$)
- infinite time evolution ($\tau \to \infty$)

*Note: In this entire lecture, $\tau$ is *not* the embedding delay.*
The Largest Lyapunov Exponent – Definition

\[ |x(t + \tau) - y(t + \tau)| = |x(t) - y(t)| \ e^{\lambda_1 \tau} \]

→ Solve for \( \lambda_1 \) and implement the limits:

First Lyapunov exponent

Let \( x \) and \( y \) be two trajectories of the dynamics.

\[
\lambda_1 := \lim_{\tau \to \infty} \lim_{|x(t) - y(t)| \to 0} \frac{1}{\tau} \ln \left( \frac{|x(t + \tau) - y(t + \tau)|}{|x(t) - y(t)|} \right)
\]

Also: largest Lyapunov exponent or just Lyapunov exponent.
Typical Evolution of a Trajectory Distance

Two identical Lorenz systems with initial conditions; one is slightly perturbed ($10^{-14}$) at $t = 30$:
Comparison with Evolution of Infinitesimal Distance

Two identical Lorenz systems with initial conditions; one is slightly perturbed \((10^{-14})\) at \(t = 30\):
Typical Evolution of a Trajectory Distance

Regimes of the average distance $D$: 

1. Alignment to the direction of largest growth: $D(\tau) \propto \sum_{i=1}^{\infty} c_i(\tau) \lambda_i^\tau$
2. Asymptotically: $c_i(\tau) \rightarrow 0$ for $i > 1$
3. Exponential growth: $D(\tau) \propto \text{diam}(\mathcal{A})$
4. Constancy on the scale of the attractor: $D(\tau) \approx \text{diam}(\mathcal{A})$
Typical Evolution of a Trajectory Distance

Regimes of the average distance $D$:

1. alignment to direction of largest growth:

$$D(\tau) \propto \sum_{i=1}^{d} c_i(\tau) \exp(\lambda_i \tau)$$

Asymptotically: $\frac{c_i(\tau)}{c_1(\tau)} \to 0$ for $i > 1$
Typical Evolution of a Trajectory Distance

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2. exponential growth:

$$D(\tau) \propto \exp(\lambda_1 \tau)$$
Typical Evolution of a Trajectory Distance

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3. constancy on the scale of the attractor:

$$D(\tau) \approx \text{diam}(\mathcal{A})$$
Translation to Time Series

• continuous trajectories  
  → discrete trajectories

• actual phase space  
  → reconstruction

• evolution of arbitrary states  
  → available trajectories

And of course: finite data, noise, ...
Wolf Algorithm
Acquisition of Instantaneous Lyapunov Exponents

1. Let $x(0)$ be the first reconstructed state.
2. Find a state $x(t_1)$ such that $|x(t_1) - x(0)| < \varepsilon$. 

\[
\hat{\lambda}_1(0) = \frac{1}{\tau} \log \frac{x(\tau) - x(t_1 + \tau)}{x(t_1) - x(t_1 + \tau)} \\
\hat{\lambda}_1(\tau) = \frac{1}{\tau} \log \frac{x(2\tau) - x(t_2 + \tau)}{x(\tau) - x(t_2 + \tau)}
\]
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**Wolf Algorithm**

**Acquisition of Instantaneous Lyapunov Exponents**

1. Let \( x(0) \) be the first reconstructed state.
2. Find a state \( x(t_1) \) such that \( |x(t_1) - x(0)| < \varepsilon \).
3. Approximate instantaneous largest Lyapunov exponent:
   \[
   \hat{\lambda}_1(0) = \frac{1}{\tau} \ln \left( \frac{x(\tau) - x(t_1 + \tau)}{x(0) - x(t_1)} \right).
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   $$\hat{\lambda}_1(0) = \frac{1}{\tau} \ln \left( \frac{x(\tau) - x(t_1 + \tau)}{x(0) - x(t_1)} \right).$$
4. Find a state $x(t_2)$ such that $|x(t_2) - x(\tau)| < \epsilon$ and $x(t_2) - x(t + \tau)$ is nearly parallel to $x(t_1 + \tau) - x(t + \tau)$. 
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5. Approximate instantaneous largest Lyapunov exponent:
   \[
   \hat{\lambda}_1(\tau) = \frac{1}{\tau} \ln \left( \frac{x(2\tau) - x(t_2 + \tau)}{x(\tau) - x(t_2)} \right).\]
Wolf Algorithm – Averaging

After acquiring the local Lyapunov exponents, estimate:

\[ \lambda_1 = \frac{1}{I - r} \sum_{i=r}^{I} \hat{\lambda}_1(i \tau) \]

The offset \( r \) ensures that the distances are aligned to the direction of largest growth.
Wolf Algorithm – Parameters

Initial distance $\epsilon$:
- too small → impact of noise too high
- too large → small region of exponential growth

Rescaling time $\tau$:
- too high → distance reaches size of attractor
- too small → small region of exponential growth
Wolf Algorithm – Problems

• Parameters have to be chosen a priori.

• Problems may be obfuscated:
  • no exponential growth due to noise
  • embedding dimension $m$ too small

• Sensitivity to noise.

• Difficult to find neighbouring trajectory segment with required properties.

→ Different way to ensure alignment to direction of largest growth.
For a given reference state $x(t)$, find all states $x(t_1), \ldots, x(t_u)$ for which $|x(t) - x(t_j)| < \varepsilon$. 

\[ x(t) \]

\[ \varepsilon \]
Rosenstein–Kantz Algorithm

1. For a given reference state $x(t)$, find all states $x(t_1), \ldots, x(t_u)$ for which $|x(t) - x(t_j)| < \varepsilon$. 
Rosenstein–Kantz Algorithm

1. For a given reference state \( x(t) \), find all states \( x(t_1), \ldots, x(t_u) \) for which \( |x(t) - x(t_j)| < \varepsilon \).

2. For a given \( \tau \), define the average distance of the respective trajectory segments from the initial one

\[
s(t, \tau) := \frac{1}{u} \sum_{j=1}^{u} |x(t + \tau) - x(t_j + \tau)|
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Rosenstein–Kantz Algorithm

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\]

3. Average over all states as reference states:

\[
S(\tau) := \frac{1}{N} \sum_{t=1}^{N} s(t, \tau)
\]
Rosenstein–Kantz Algorithm

1. For a given *reference* state \( x(t) \), find all states \( x(t_1), \ldots, x(t_u) \) for which \( |x(t) - x(t_j)| < \varepsilon \).

2. For a given \( \tau \), define the average distance of the respective trajectory segments from the initial one

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3. Average over all states as reference states:

\[
S(\tau) := \frac{1}{N} \sum_{t=1}^{N} S(t, \tau)
\]

4. Obtain \( \lambda_1 \) from region of exponential growth of \( S(\tau) \).
Rosenstein–Kantz Algorithm – Example

(Different lines correspond to different $\epsilon$ and $m$.)

Rosenstein–Kantz Algorithm – Mind How You Average

1. Average over the neighbourhood of a reference state → \( s(t, \tau) \).

2. Average \( s(t, \tau) \) over all reference states → \( S(\tau) \).

3. Obtain \( \lambda_1 \) from slope of \( S(\tau) \).

Density of states in a region of the attractor affects:
• reference states
• states in neighbourhood of given reference state

Separating the averaging in steps 1 and 2 (instead of averaging of all pairs closer than \( \varepsilon \)) ensures that density is accounted for only once (and not twice).
Rosenstein–Kantz Algorithm – Advantages and Problems

- Region of exponential growth can be determined a posteriori. Be careful of wishful thinking though.
- Absence of exponential growth usually detectable (but only usually).
- Region of strong noise influence can be detected and excluded.
- Can only determine the largest Lyapunov exponent.
extensions and alternatives

• tangent-space methods
  → require estimate of Jacobian

• further Lyapunov exponents
  → requires a lot of data
**Lyapunov Spectrum and Types of Dynamics**

For bounded, continuous-time dynamical systems:

<table>
<thead>
<tr>
<th>signs of Lyapunov exponents</th>
<th>Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>-, --, ---, ...</td>
<td>fixed point</td>
</tr>
<tr>
<td>+, ++, ++++, ..., +0, ++0, ...</td>
<td>not possible (unbounded)</td>
</tr>
<tr>
<td>0, 00, 000, ...</td>
<td>no dynamics ($f = 0$)</td>
</tr>
<tr>
<td>0−, 0−−, 0−---, ...</td>
<td>periodic / limit cycle</td>
</tr>
<tr>
<td>00−, 00--0, 00---, ...</td>
<td>quasiperiodic (torus)</td>
</tr>
<tr>
<td>000−, 0000−, ..., 000--0, ...</td>
<td>quasiperiodic (hypertorus)</td>
</tr>
<tr>
<td>0000−, 00000−, ..., 0000--0, ...</td>
<td></td>
</tr>
<tr>
<td>+0−, +0--0, +0---0, ...</td>
<td>chaos</td>
</tr>
<tr>
<td>++0−, +++0−, ..., ++0--0, ...</td>
<td>hyperchaos</td>
</tr>
<tr>
<td>∞, ...</td>
<td>noise</td>
</tr>
</tbody>
</table>
Interpretation

• Stability and type of the dynamics:
  \( \lambda_1 > 0 \)  chaos, instable dynamics
  \( \lambda_1 = 0 \)  regular dynamics
  \( \lambda_1 < 0 \)  fixed-point dynamics

• Quantification of loss of information.

• Prediction horizon:

\[ \tau_p \approx \frac{-\ln(\rho)}{\sum_{i,\lambda_i > 0} \lambda_i} \]

• \( \rho \): Accuracy of measurement (initial state).
• \( \sum_{i,\lambda_i > 0} \): Sum of positive Lyapunov exponents.