



Dimension

- Charakterisierung eines geometrischen Objekts
- euklidische Geometrie: Dimension ganzzahlig

Objekt		Dimension
Punkt	.	0
Linie	—	1
Fläche		2
Kubus		3
n -Kubus		n

Dimension: Anzahl der zur Charakterisierung eines geometrischen Objekts notwendigen Freiheitsgrade

Zeitreihenanalyse: Zahl der für Modellierung eines physikalischen Systems notwendigen Gleichungen

Dimension

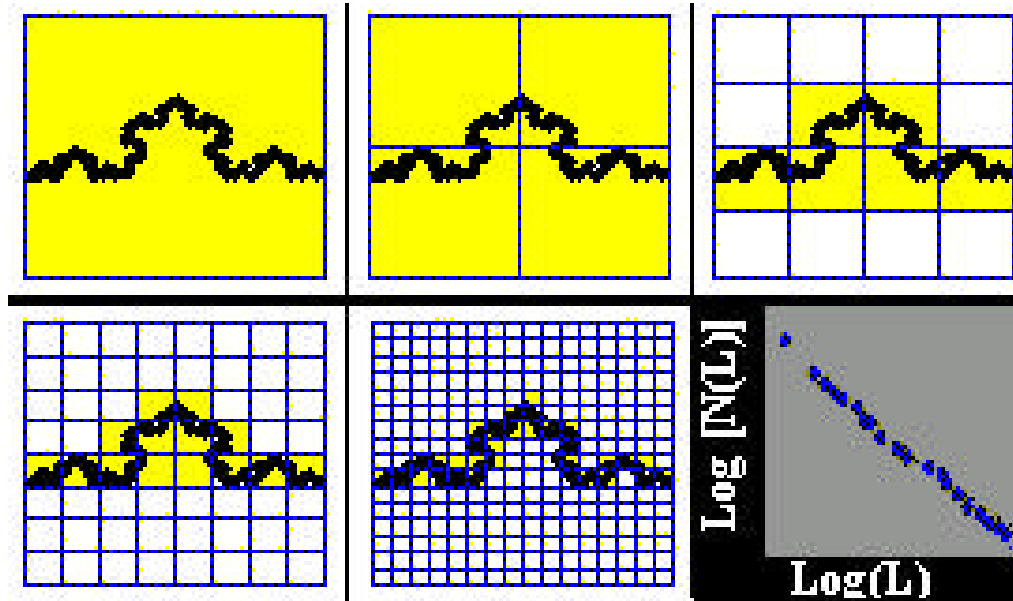
Generalisierung auf nicht-euklidische Geometrien: **Hausdorff-Dimension**
(F. Hausdorff, 1919)

- Dimension eines (nicht-)euklidischen Objekts im m -dimensionalen Raum
 - > überdecke Objekt mit m -dimensionalen Hyperkuben der Kantenlänge ε
 - > ermittle minimale Anzahl $N(\varepsilon)$ der Hyperkuben, die zur vollständigen Überdeckung notwendig sind

$$\Rightarrow N(\varepsilon) \underset{(\varepsilon \rightarrow 0)}{\infty} \varepsilon^{-D_0}$$

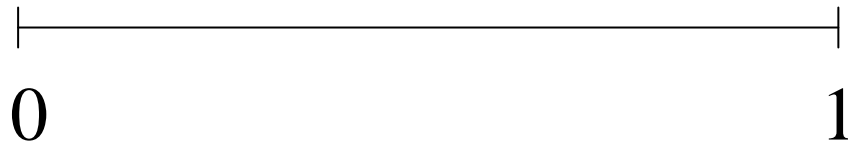
D_0 Hausdorff-Dimension (fraktale -, Box- oder Überdeckungsdimension)

box-counting

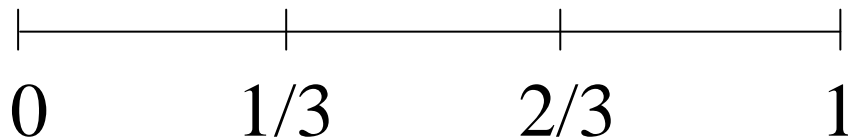


Hausdorff-Dimension

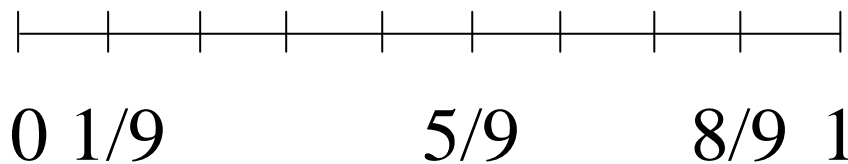
Hausdorff-Dimension einer Linie



$$\varepsilon = 1, N = 1$$



$$\varepsilon = 1/3, N = 3$$



$$\varepsilon = 1/9, N = 9$$

$$D_0 = \frac{\log[N(\varepsilon) / N(\varepsilon')]}{\log(\varepsilon / \varepsilon')} = \frac{\log 3}{\log 3} = 1$$

Hausdorff-Dimension

seltsame Attraktoren

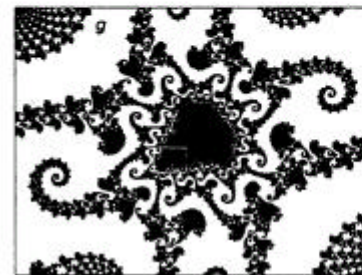
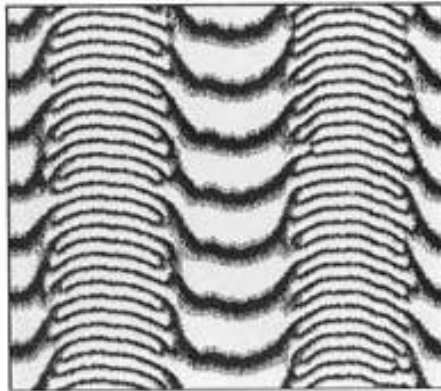
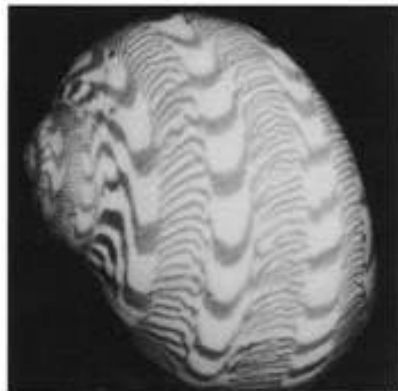
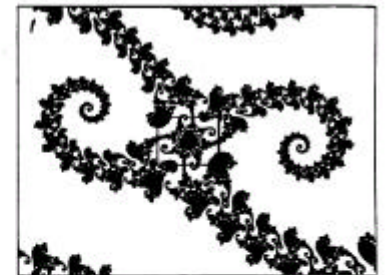
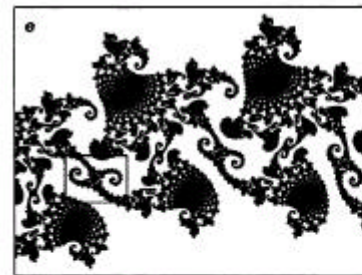
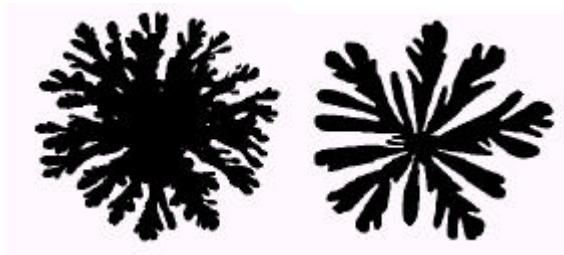
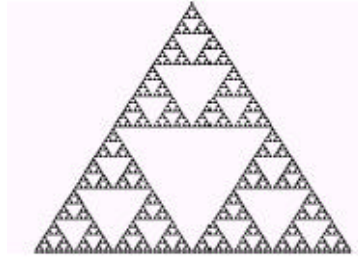
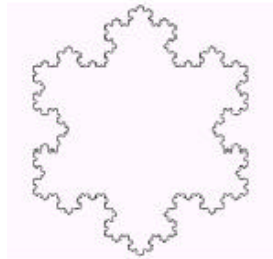
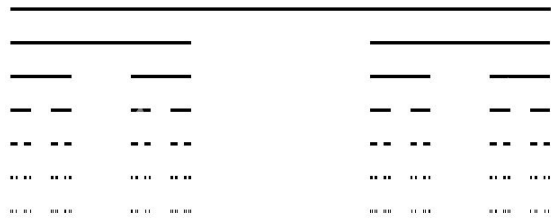
- seltsam, aufgrund der sensitiven Abhängigkeit von den Anfangsbedingungen
- seltsames Zeitverhalten \leftrightarrow seltsame Attraktorgeometrie
- aufgrund der Dissipation
Stauchen, Strecken und Falten (nichtlinear !) in verschiedenen Raumrichtungen
- führt zu *Selbstähnlichkeit* (im statistischen Sinne) und zur *Skaleninvarianz* (Struktur auf beliebigen Skalen)

Fraktal

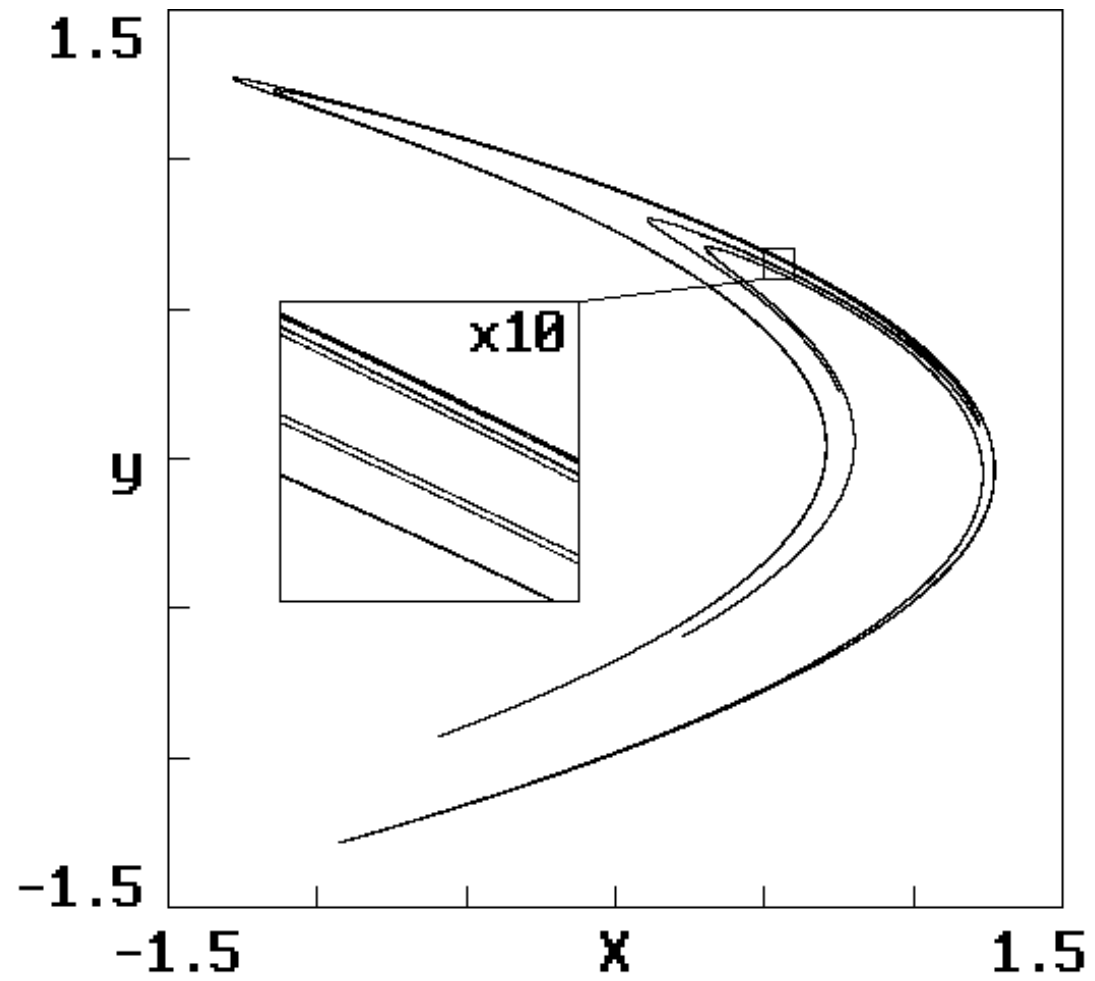
Def.: Eine Menge F ist ein *Fraktal*, wenn

- F eine Feinstruktur hat
- F irregulär ist
- F Selbstähnlichkeit aufweist
- die fraktale Dimension größer als die topologische Dimension ist

Fraktale



Fraktale



Fraktale

>> Physics News Update, 92, 19 August 1992

The landscape of DNA may be fractal

(Phys Rev Lett 22 Jun 92)

>> Physics News Update, 353, 5 Januar 1998

Fractal patterns inside cells can reveal breast cancer

(Phys Rev Lett, 12 Jan 98)

>> Physics News Update, 399, 26 Oktober 1998

Tumor growth can be fractal (fractal dimension: 1.21)

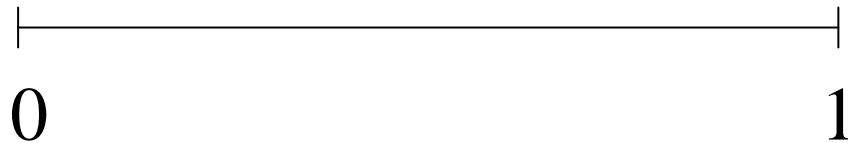
(Phys Rev Lett, 2 Nov 98)

Cantor Menge



Hausdorff-Dimension

Hausdorff-Dimension eines Fraktals (z.B. Cantor-Menge)



$$\varepsilon = 1, N = 1$$



$$\varepsilon = 1/3, N = 2$$



$$\varepsilon = 1/9, N = 4$$

$$D_0 = \frac{\log[N(\varepsilon) / N(\varepsilon')]}{\log(\varepsilon / \varepsilon')} = \frac{\log 2}{\log 3} = 0,6309$$

Hausdorff-Dimension

Hausdorff-Dimension (fraktale Dimension) der Cantor-Menge:

$$D_0 = \frac{\log[N(\varepsilon) / N(\varepsilon')]}{\log(\varepsilon / \varepsilon')} = \frac{\log 2}{\log 3} = 0,6309$$

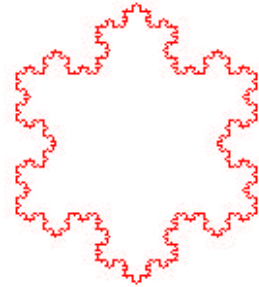
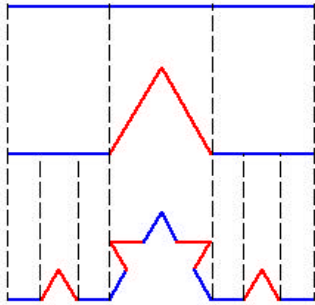
Länge (topologische Dimension) der Cantor-Menge:

$$L = 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} \dots = 1 - \frac{1}{3} \sum_{v=0}^{\infty} \left(\frac{2}{3}\right)^v = 0$$

$$D_0 > L$$

Hausdorff-Dimension

Hausdorff-Dimension eines Fraktals (z.B. Kochsche Schneeflocke)



nach 1. Iterationsschritt:
 $\varepsilon = 1/3, N = 4$

nach k. Iterationsschritt:
 $\varepsilon = (1/3)^k, N = 4^k$

unendlich zackige Kurve, unendlich lang, an **keiner** Stelle differenzierbar, aber umschließt ein **endliches** Flächenstück (wie lang ist die Küste von England ?)

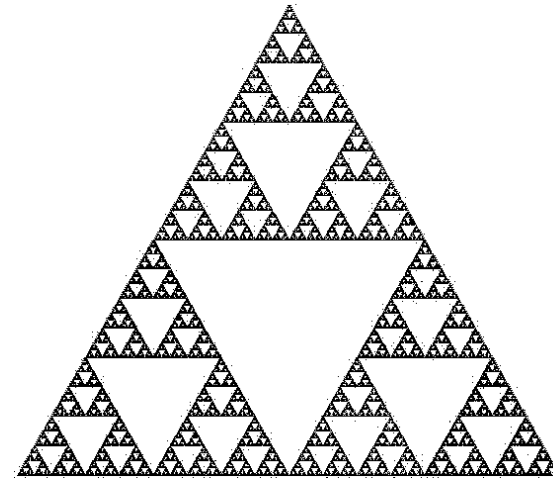
$$D_0 = \frac{\log[N(\varepsilon) / N(\varepsilon')]}{\log(\varepsilon / \varepsilon')} = \frac{\log 4}{\log 3} = 1,2618$$

Hausdorff-Dimension

Hausdorff-Dimension eines Fraktals

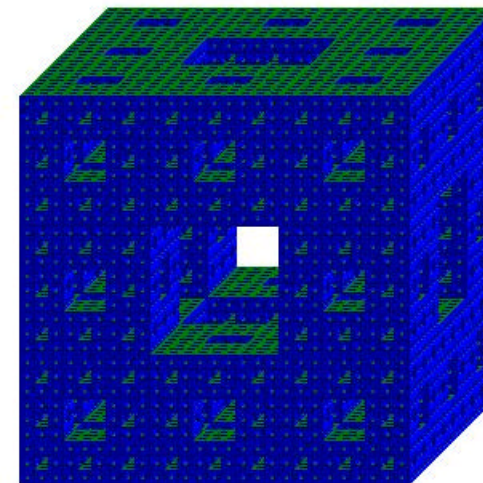
Pfeilspitzenkurve

$$D_0 = 1,5849$$



Sierpinski Schwamm

$$D_0 = 2,7268$$



Hausdorff-Dimension

Hausdorff-Dimension eines Fraktals

Logistische Abbildung
($r=3,5699$)

$$D_0 = 0,538$$

Hénon-Abbildung
($a=1,4; b=0,3$)

$$D_0 = 1,26$$

Zaslavskii-Abbildung
 $\left(\Gamma = 3; v = \frac{400}{3}; \varepsilon = 0,3; \mu = \frac{1 - e^{-\Gamma}}{\Gamma} \right)$

$$D_0 = 1,39$$

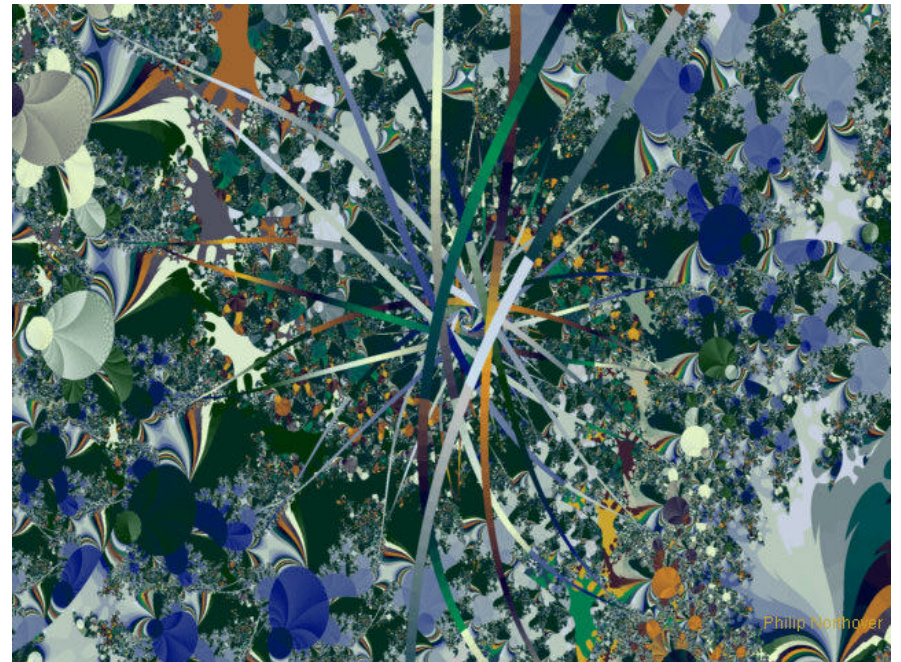
Lorenz-System
($R=28, s=10, b=8/3$)

$$D_0 = 2,06$$

Jack the dripper: chaos in modern art

Of all the abstract expressionist painters, Jackson Pollock was perhaps the most controversial. He would dash around large canvases rolled out on the floor of his barn, dripping paint from a wooden stick. The critics poured scorn on his paintings, calling them "meaningless chaos". But chaos is now a rigorous scientific concept that we know appears throughout nature. One important part of chaos theory is fractal behaviour, which describes objects that have similar patterns when viewed at different magnifications. Richard Taylor, a physicist at the University of New South Wales, has now discovered this characteristic in many of Pollock's works. Rather than being the fraud that many people assume, Taylor believes that Pollock subconsciously understood the patterns of nature so well that he was able to capture their very essence -- chaos and fractals -- on canvas.

Physics Web November 1997



Physics News: June 1999

Fractals determine date of paintings

[4 Jun 1999] Paintings by the late Jackson Pollock - considered to be one of the fathers of modern art - can be dated by fractal geometry according to Australian physicists (Nature 399 422). Pollock's artwork during the late 1940s consisted of paint dripped from a can onto large canvases spread out on the floor of his barn. Richard Taylor, Adam Micolich and David Jonas from the University of New South Wales in Sydney discovered that ***the fractal dimension of Pollock's drip paintings increased from nearly 1.0 in 1943, to 1.72 in 1952***, suggesting that Pollock gradually refined his technique over to time to make his painting more fine grained.

verallgemeinerte Dimensionen

Hausdorff-Dimension D_0 hochdimensionaler Systeme mittels *box-counting* i.A. schwer zu bestimmen; betrachte verallgemeinerte Dimensionen D_k :

Partitionierung des m -dimensionalen Phasenraums mit M Hyperkuben der Kantenlänge ε ($\varepsilon \rightarrow 0$);

betrachte Wahrscheinlichkeit p_i , einen Punkt des Attraktors in Hyperkubus i ($i=1, \dots, M(\varepsilon)$) zu finden:

$$p_i = \lim_{N \rightarrow \infty} \frac{N_i}{N} \quad \begin{array}{l} N_i = \text{Anzahl der Trajektorienpunkte in Zelle } i \\ N = \text{Gesamtanzahl der Trajektorienpunkte} \end{array}$$

$$\Rightarrow D_k = \lim_{\varepsilon \rightarrow 0} \frac{1}{k-1} \frac{\log \left(\sum_{i=1}^{M(\varepsilon)} p_i^k \right)}{\log \varepsilon} \quad k = 0, 1, 2, \dots, \infty$$

verallgemeinerte Dimensionen

$k \rightarrow 0$ $D_0 =$ Hausdorff-Dimension

$$D_k = \lim_{\varepsilon \rightarrow 0} \frac{1}{k-1} \frac{\log \left(\sum_{i=1}^{M(\varepsilon)} p_i^k \right)}{\log \varepsilon}$$



$$D_0 = - \lim_{\varepsilon \rightarrow 0} \frac{\log M(\varepsilon)}{\log \varepsilon}$$

D_0 zählt die Anzahl der nichtleeren Hyperkuben

verallgemeinerte Dimensionen

$k \rightarrow 1$ (Regel von l'Hospital) $D_1 =$ Informations-Dimension

$$D_k = \lim_{\varepsilon \rightarrow 0} \frac{1}{k-1} \frac{\log \left(\sum_{i=1}^{M(\varepsilon)} p_i^k \right)}{\log \varepsilon}$$



$$D_1 = - \lim_{\varepsilon \rightarrow 0} \frac{\log S(\varepsilon)}{\log \varepsilon}$$

mit

Shannon-Entropie:

$$S(\varepsilon) = - \sum_{i=0}^{M(\varepsilon)} p_i \log p_i$$

D_1 beschreibt Informationszuwachs, einen Trajektorienpunkt in Hyperkubus i zu finden

verallgemeinerte Dimensionen

D_1 = Informations-Dimension

Charakterisierung der *Homogenität* eines Attraktors:

bei homogenen Attraktoren: $p_i = \frac{1}{M(\varepsilon)}$ in allen Hyperkuben i

$$\Rightarrow S(\varepsilon) = \log M(\varepsilon)$$

$$\Rightarrow D_1 = D_0$$

$|D_0 - D_1|$ Maß für Inhomogenität eines Attraktors

verallgemeinerte Dimensionen

im allgemeinen gilt: $D_{k'} \leq D_k \quad k' > k$

- **statisches** Maß der Attraktor-Eigenschaften
- Anzahl der Freiheitsgrade eines Systems
- Maß für Systemkomplexität

reguläre Dynamik
chaotische Dynamik
stochastische Dynamik



D ganzzahlig
D fraktal
D $\rightarrow \infty$

verallgemeinerte Dimensionen

$k \rightarrow 2$ $D_2 =$ Korrelations-Dimension

$$D_k = \lim_{\varepsilon \rightarrow 0} \frac{1}{k-1} \frac{\log \left(\sum_{i=1}^{M(\varepsilon)} p_i^k \right)}{\log \varepsilon}$$



$$D_2 = \lim_{\varepsilon \rightarrow 0} \frac{\log C(\varepsilon)}{\log \varepsilon}$$

mit $C(\varepsilon) =$ Korrelationssumme (Grassberger & Procaccia, 1983)

Korrelationssumme

quadrierte Wahrscheinlichkeiten für $k \rightarrow 2$

betrachte Relation zur diskreten
Korrelationsfunktion

$$C_{vv}(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N v(i)v(i+\tau)$$

zeitverzögerte Phasenraumvektoren (Takens-Einbettung):

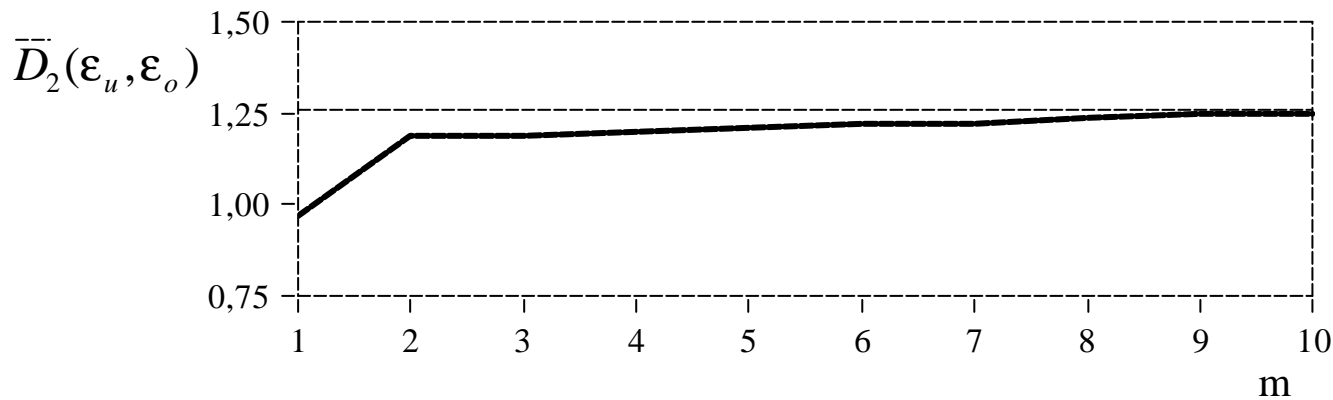
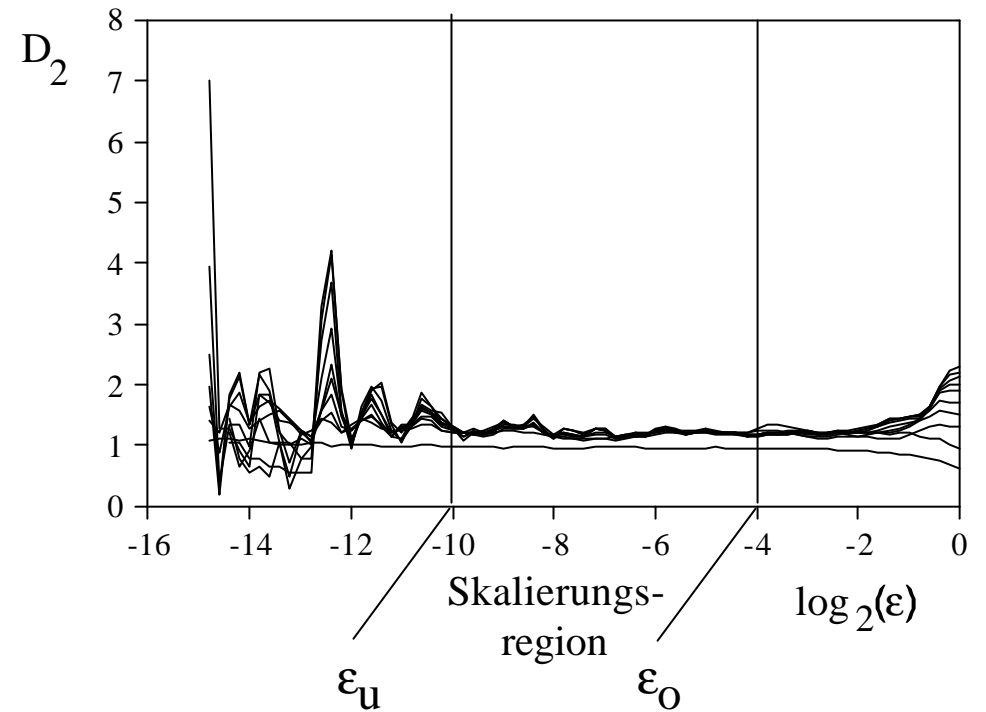
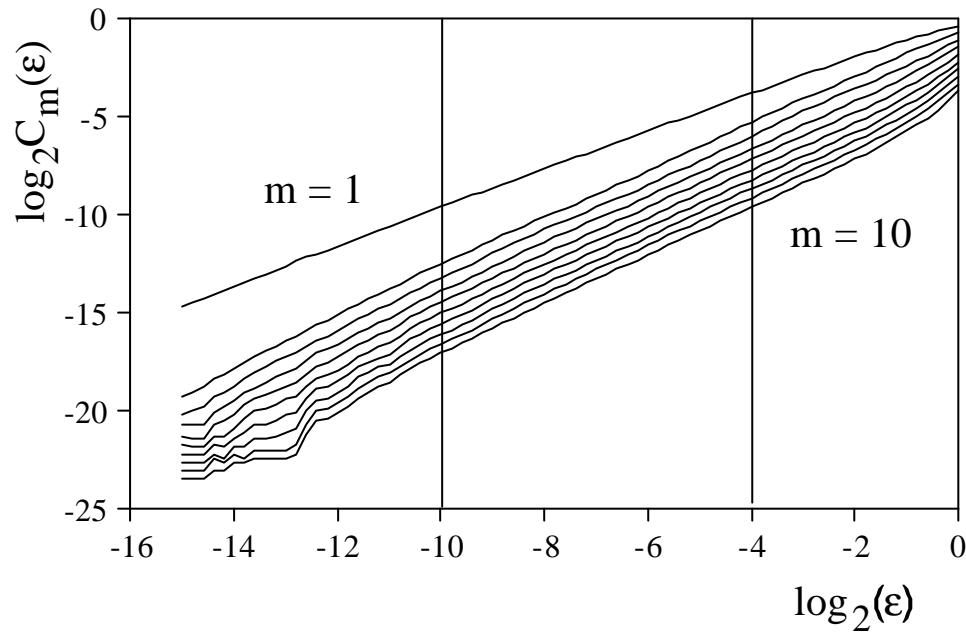
$$\mathbf{v}(i) = (v(i), v(i-\tau), \dots, v(i-(m-1)\tau))$$

Korrelationssumme:

$$C_m(\varepsilon) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^N H(\varepsilon - |\mathbf{v}_m(i) - \mathbf{v}_m(j)|)$$

$$H = \text{Heaviside-Funktion} \begin{cases} H(a) = 0, a < 0 \\ H(a) = 1, \text{sonst} \end{cases}$$

Abschätzung der Korrelationsdimension

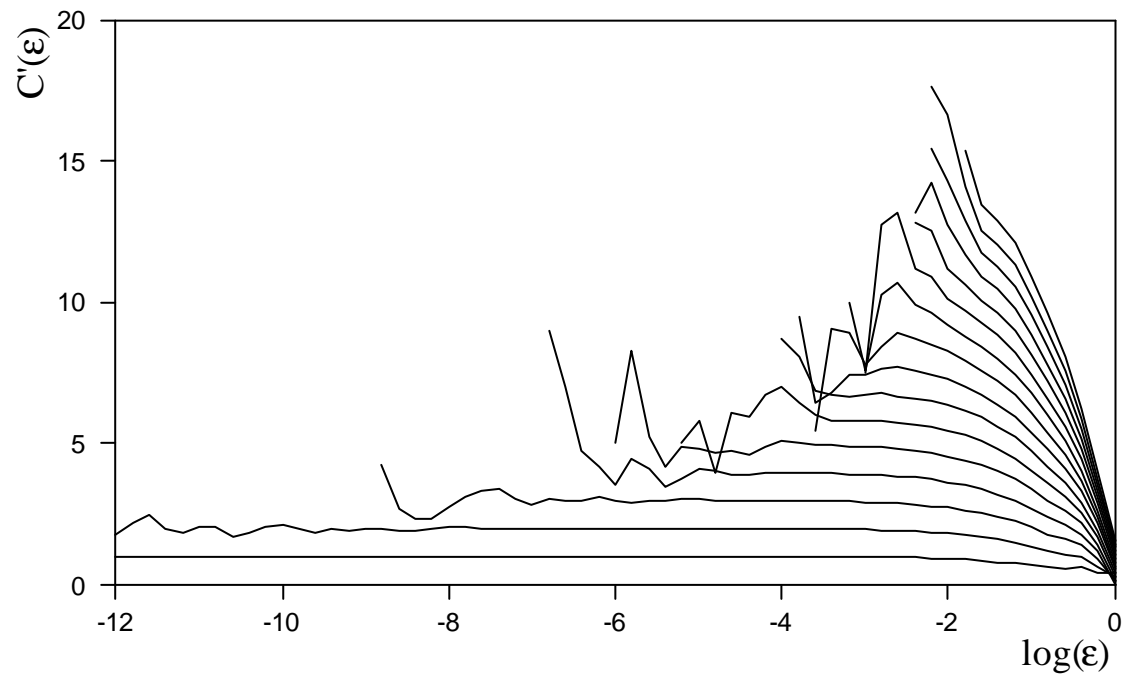
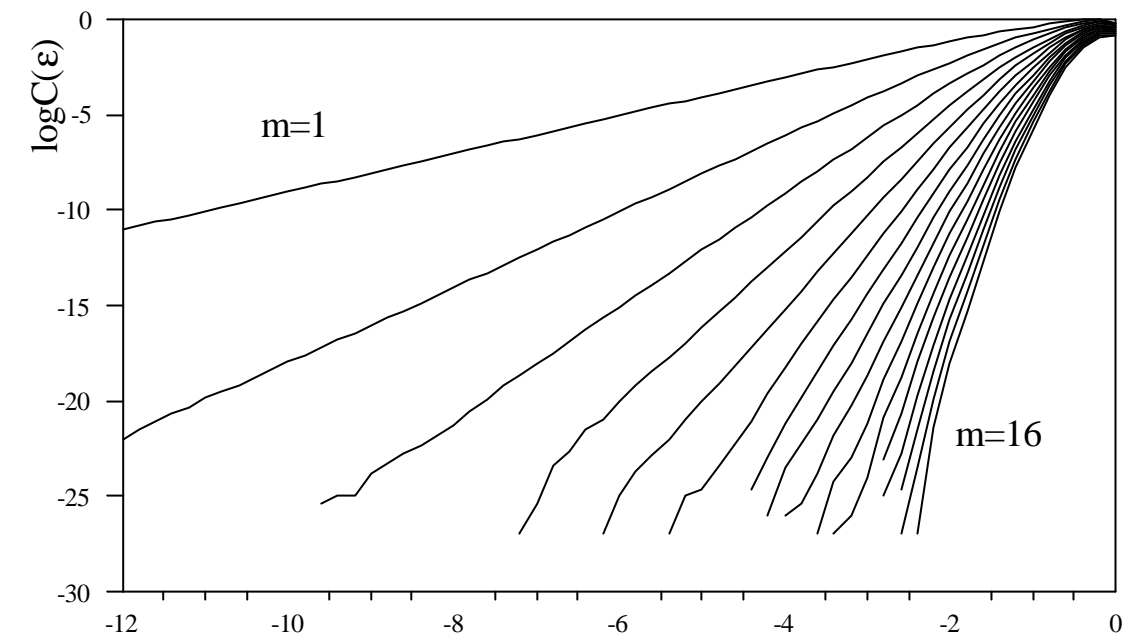


Hénon-Abbildung:

$a = 1,4; b = 0,3$

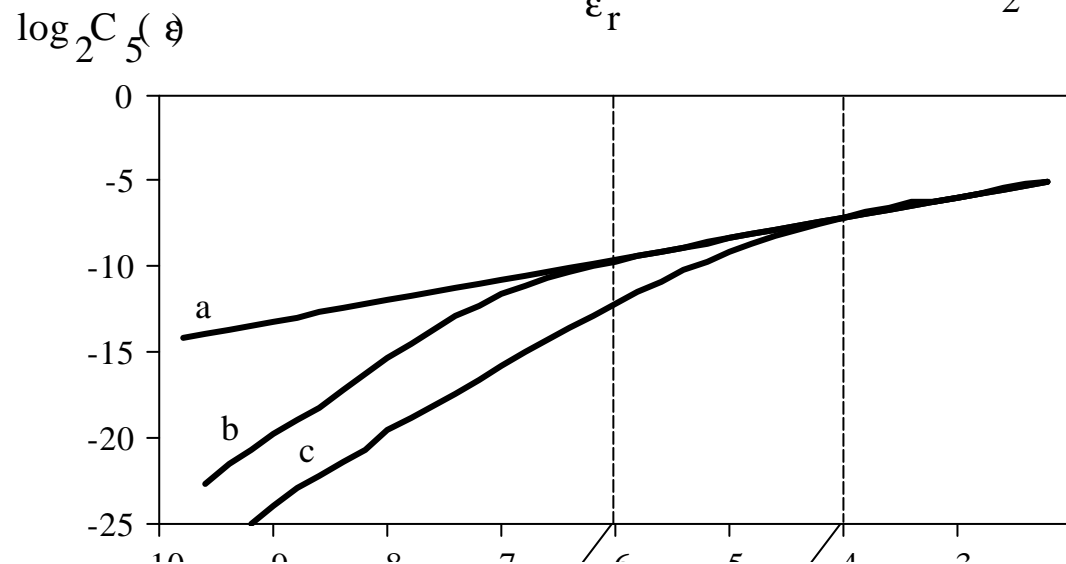
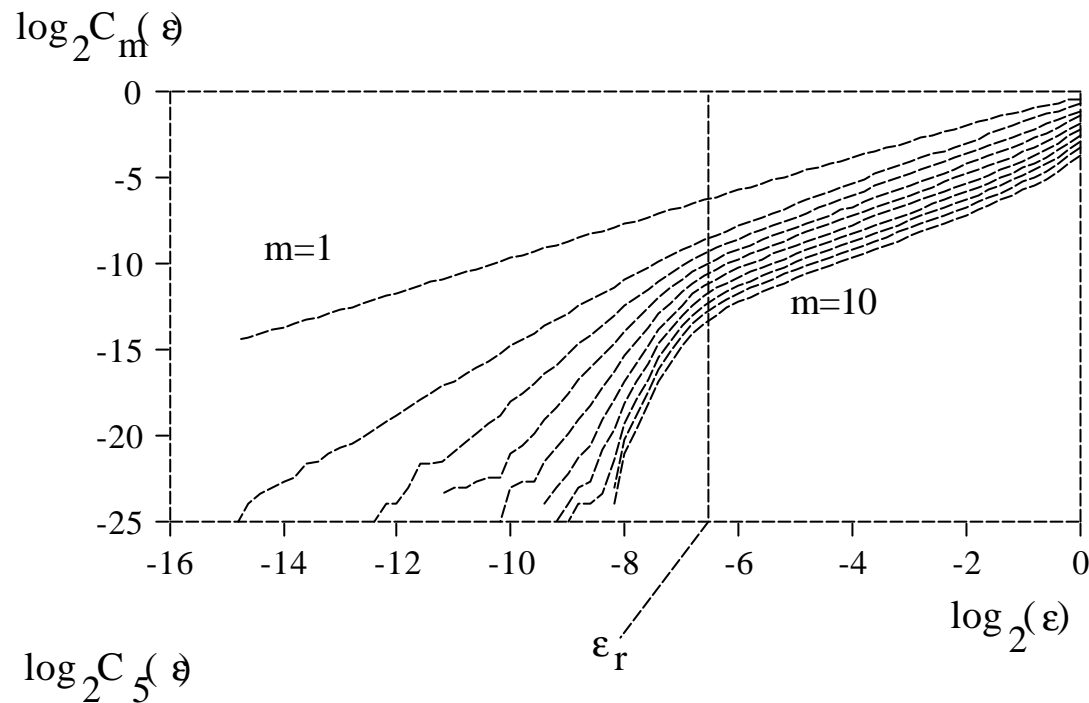
$N = 8192$

$D_2^{theor.} = 1,26$



Weisses Rauschen, 16K Daten

Korrelationssumme und Rauschamplitude



Hénon (a)

+
weißes Rauschen

(b) Amplitude r1

(c) Amplitude r2